Orthogonality of Matrices

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Abstract
Let $A$ and $B$ be rectangular matrices. Then $A$ is orthogonal to $B$ if
$$\|A + \mu B\| \geq \|A\| \quad \text{for every scalar } \mu.$$ Some approximation theory and convexity results on matrices are used to study orthogonality of matrices and answer an open problem of Bhatia and Šemrl.

1 Introduction

Let $(\mathbb{F}^{m \times n}, \| \cdot \|)$ be a normed matrix space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Suppose $A, B \in \mathbb{F}^{m \times n}$, we say that $A$ is orthogonal to $B$ (in the Birkhoff-James sense [4]) if
$$\|A + \mu B\| \geq \|A\| \quad \text{for every } \mu \in \mathbb{F}.$$ The above condition can be interpreted in the context of approximation theory as follows. Suppose $A \in \mathbb{F}^{m \times n}$ is not in the linear subspace $\mathcal{W}$ spanned by the matrix $B \in \mathbb{F}^{m \times n}$. Then the zero matrix is the best approximation to $A$ among all matrices in $\mathcal{W}$. In this note, we use some approximation theory and convexity results in matrix spaces to study orthogonality of matrices. Our results cover and extend those of other authors [1, 5].

We collect some preliminary results in Section 2, and use them to characterize matrix pairs which are orthogonal with respect to Schatten $p$-norms in Section 3. In the last section, we study orthogonal matrix pairs with respect to operator norms and give a counter-example to a conjecture of Bhatia and Šemrl [1].

We always assume that $\mathbb{F}^{m \times n}$ is equipped with the inner product $(A, B) = \text{tr}(AB^*)$. This includes the special case when $\mathbb{F}^{m \times 1} = \mathbb{F}^n$ and $(x, y) = \text{tr}(xy^*) = y^*x$. Denote by $\{e_1, \ldots, e_n\}$ the standard basis for $\mathbb{F}^n$, and $\{E_{11}, E_{12}, \ldots, E_{nn}\}$ the standard basis for $\mathbb{F}^{m \times n}$. Let $U_n(\mathbb{F})$ be the unitary or orthogonal group depending on $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$.

For notational convenience, we always consider $m \times n$ matrix with $m \leq n$ in our discussion; the case $m > n$ can be treated similarly. For $A \in \mathbb{F}^{m \times n}$, denote by $s_1(A) \geq \cdots \geq s_m(A)$ the singular values of $A$, which are the nonnegative square roots of the eigenvalues of the matrix $AA^*$. We always use the fact that every matrix $A \in \mathbb{F}^{m \times n}$ has a singular value decomposition, viz., $A = U^*(\sum_{j=1}^m s_j(A)E_{jj})V$ for some $U \in U_m(\mathbb{F})$ and $V \in U_n(\mathbb{F})$.

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2 Preliminary Results

Let $\| \cdot \|$ be a norm on $\mathbb{F}^{m \times n}$. The dual norm of $\| \cdot \|$ is defined by

$$\| X \|^D = \max \{ \| (X, Y) \| : \| Y \| \leq 1 \}.$$ 

We have the following result, which is a special case of the general theorem of Singer in [9, p.170].

**Proposition 2.1** Let $\| \cdot \|$ be a norm on $\mathbb{F}^{m \times n}$. Suppose $A, B \in \mathbb{F}^{m \times n}$ are such that $A$ is not a multiple of $B$. Then

$$\| A + \mu B \| \geq \| A \| \quad \text{for all } \mu \in \mathbb{F}$$

if and only if there exist $h$ extreme points $F_1, \ldots, F_h \in \mathbb{F}^{m \times n}$ of the unit ball $\{ Y \in \mathbb{F}^{m \times n} : \| Y \|^D \leq 1 \}$ in the dual space $(\mathbb{F}^{m \times n}, \| \cdot \|^D)$ with $h \leq 3$ in the complex case and $h \leq 2$ in the real case, and positive numbers $t_1, \ldots, t_h$ with $t_1 + \cdots + t_h = 1$ such that

$$\sum_{j=1}^{h} t_j (F_j, B) = 0 \quad \text{and} \quad (F_j, A) = \| A \|, \quad j = 1, \ldots, h. \quad (1)$$

The numerical range of a matrix $A \in \mathbb{F}^{n \times n}$ is defined by

$$W(A) = \{ x^* Ax : x \in \mathbb{F}^n, \ x^* x = 1 \},$$

which has been studied extensively, see [3, Chapter 1]. We have the following result.

**Proposition 2.2** Let $A \in \mathbb{F}^{n \times n}$. Then $W(A)$ is convex.

**Proof.** For the complex case, see [3, Chapter 1]. For the real case, note that $W(A)$ can be viewed as the image of the unit sphere in $\mathbb{R}^n$ under the continuous map $x \mapsto x^* Ax$. Since the unit sphere in $\mathbb{R}^n$ is a compact connected set, the set $W(A)$ is a closed interval. \qed

3 The Schatten $p$-Norms

Suppose $1 \leq p \leq \infty$. The Schatten $p$-norm of $A \in \mathbb{F}^{m \times n}$ is defined by

$$S_p(A) = \begin{cases} \left\{ \sum_{j=1}^{n} s_j(A)^p \right\}^{1/p} & \text{if } 1 \leq p < \infty, \\ \max\{s_j(A) : 1 \leq j \leq m\} & \text{if } p = \infty. \end{cases}$$

We refer the readers to [7] for basic properties of the Schatten $p$-norms. Here we characterize $A \in \mathbb{F}^{m \times n}$ which are orthogonal to a given matrix $B \in \mathbb{F}^{m \times n}$ with respect to the Schatten $p$-norms. We shall use the basic fact that the dual space of $(\mathbb{F}^{m \times n}, S_p)$ is $(\mathbb{F}^{m \times n}, S_q)$, where
\[1/p + 1/q = 1.\] Moreover, \(F \in \mathbb{F}^{m \times n}\) is an extreme point of the unit norm ball of \((\mathbb{F}^{m \times n}, S_p)\) if and only if

(i) \(p = 1\) and \(F = xy^*\) for some unit vectors \(x \in \mathbb{F}^m\) and \(y \in \mathbb{F}^n\);

(ii) \(1 < p < \infty\) and \(S_p(F) = 1\);

(iii) \(p = \infty\) and \(FF^* = I_m\).

In [1] (see also [5]) the authors obtained results for complex square matrices with \(p > 1\) and partial results for \(p = 1\) by different methods.

**Theorem 3.1** Let \(A, B \in \mathbb{F}^{m \times n}\), where \(m \leq n\). The following conditions are equivalent.

(a) \(S_\infty (A + \mu B) \geq S_\infty (A)\) for all \(\mu \in \mathbb{F}\).

(b) There exist unit vectors \(x \in \mathbb{F}^m\) and \(y \in \mathbb{F}^n\) such that \(S_\infty (A) = x^*Ay\) and \(x^*By = 0\), equivalently, there is a unit vector \(y \in \mathbb{F}^n\) such that \(S_\infty (A) = l_2(Ay)\) and \((Ay, By) = 0\).

(c) For any \(U \in \mathbb{F}^{m \times k}\) with orthonormal columns that form a basis for the eigenspace of \(AA^*\) corresponding to the largest eigenvalue, and \(V = A^*U/S_\infty (A) \in \mathbb{F}^{n \times k}\), we have

\[0 \in W(U^*BV)\quad \text{or} \quad 0 \in W(U^*BA^*U).\]

**Proof.** To prove the theorem, we use Proposition 2.1 with \(\| \cdot \| = S_\infty\), and the fact that \((\mathbb{F}^{m \times n}, S_1)\) is the dual space of \((\mathbb{F}^{m \times n}, S_\infty)\).

Suppose (a) holds. By Proposition 2.1, there exist extreme points \(F_j = x_jy_j^* \in \mathbb{F}^{m \times n}\) of the unit ball of \((\mathbb{F}^{m \times n}, S_1)\) with \(1 \leq j \leq h\), and some positive constants \(t_1, \ldots, t_h\) with \(t_1 + \cdots + t_h = 1\) so that \((F_j, A) = S_\infty (A)\) for \(j = 1, \ldots, h\), and \((\sum_{j=1}^h t_j F_j, B) = 0\). By our assumption on \(U\), for each \(j = 1, \ldots, h\), there is a unit vector \(v_j \in \mathbb{F}^k\) so that \(x_j = U v_j\), and \(y_j = V v_j\). Thus,

\[0 = (\sum_{j=1}^h t_j F_j, B) = (\sum_{j=1}^h t_j v_j^* (U^*BV) v_j),\]

which is an element in the convex hull of \(W(U^*BV)\), equivalently, \(0 \in W(U^*BV)\) by Proposition 2.2. By the fact that \(A^*U = S_\infty (A) V\), we see \(0 \in W(U^*BV)\) if and only if \(0 \in W(U^*BA^*U)\). Hence, condition (c) holds.

If (c) holds, and \(v \in \mathbb{F}^k\) is a unit vector such that \(0 = v^* U^* BA^* U v\), then \(x = U v\) and \(y = A^* x / S_\infty (A)\) are the unit vectors satisfying (b).

If (b) holds, then \(F_1 = xy^* \in \mathbb{F}^{m \times n}\) is an extreme point of the unit ball of \((\mathbb{F}^{m \times n}, S_1)\); see (i). So condition (1) holds with \(h = 1\). By Proposition 2.1, condition (a) holds. \(\square\)

Note that condition (b) in the above theorem looks simpler than (c) as it does not depend on the construction of a basis for the eigenspace of \(AA^*\). Nonetheless, in practice, it is easier to check condition (c) by studying \(W(U^*BA^*U)\).
Theorem 3.2 Let $1 < p < \infty$, $m \leq n$. Suppose $A, B \in F^{m \times n}$, where $A = HX$ for some positive semi-definite $H \in F^{m \times m}$ and $X \in F^{m \times n}$ with $XX^* = I_m$. Then

$$S_p(A + \mu B) \geq S_p(A) \quad \text{for all } \mu \in F$$

if and only if for any $U \in U_m(F)$ and $V \in U_n(F)$ satisfying $UAV^* = \sum_{j=1}^m s_j(A)E_{jj}$ we have

$$\text{tr} \left[ U^* \left( \sum_{j=1}^m s_j(A)E_{jj} \right) V B^* \right] = 0,$$

equivalently, $\text{tr} \left( H^{p-1} X B^* \right) = 0$.

Proof. The theorem readily follows from Proposition 2.1 and the fact that if $U \in U_m(F)$ and $V \in U_n(F)$ are such that $A = U^* (\sum_{j=1}^m a_j E_{jj}) V$ with $a_1 \geq \cdots \geq a_m \geq 0$, then $F = \gamma^{-1} U^* (\sum_{j=1}^m a_j^{p-1} E_{jj}) V$ with

$$\gamma = \left\{ \sum_{j=1}^m a_j^{p-1} \right\}^{1/q} \left\{ \sum_{j=1}^m a_j^{q(1-1/p)} \right\}^{1/q} \left\{ \sum_{j=1}^m a_j^q \right\}^{1/q}$$

is the unique extreme point of the dual norm ball in the dual space of $(F^{m \times n}, S_p)$ satisfying $(F, A) = S_p(A)$. \hfill \Box

Theorem 3.3 Let $A, B \in F^{m \times n}$, where $m \leq n$. The following conditions are equivalent.

(a) $S_1(A + \mu B) \geq S_1(A)$ \quad for all $\mu \in F$.

(b) There exists $F \in F^{m \times n}$ such that $S_\infty(F) \leq 1$, $\text{tr} (AF^*) = S_1(A)$ and $\text{tr} (BF^*) = 0$.

(c) For any $U \in U_m(F)$ and $V \in U_n(F)$ satisfying $UAV^* = \sum_{j=1}^m s_j(A)E_{jj}$, we have

$$UBV^* = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \quad \text{and} \quad |\text{tr}(B_{11})| \leq S_1(B_{22}),$$

where $B_{11}$ is $k \times k$ with $k = \text{rank}(A)$, and by convention $S_1(B_{22}) = 0$ if $m = k$.

Proof. Suppose (a) holds. By Proposition 2.1 and (iii), there exist extreme points $F_1, \ldots, F_h \in F^{m \times n}$ satisfying $F_j F_j^* = I_m$, where $1 \leq h \leq 3$, and positive constants $t_1, \ldots, t_h$ with $t_1 + \cdots + t_h = 1$ such that $S_1(A) = (F_j, A)$ for $j = 1, \ldots, h$, and $(\sum_{j=1}^h t_j F_j, B) = 0$. Then condition (b) holds with $F = \sum_{j=1}^h t_j F_j$.

Suppose (b) holds. Let $U \in U_m(F)$ and $V \in U_n(F)$ satisfy $A = U^* (\sum_{j=1}^m s_j(A)E_{jj}) V$. Furthermore, assume that $\text{rank}(A) = k$. Since $\text{tr} (AF^*) = S_1(A)$, we see that

$$U F V^* = \begin{pmatrix} I_k & 0 \\ 0 & G \end{pmatrix},$$

where $S_\infty(G) \leq 1$. Thus,

$$0 = \text{tr} (BF^*) = \text{tr} \left[ \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} I_k & 0 \\ 0 & G \end{pmatrix} \right]^*$$

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implies that 
\[ |\text{tr} (B_{11})| = |\text{tr} (B_{22}G^*)| \leq S_1(B_{22}), \]
i.e., condition (c) holds.

Finally, if (c) holds with \( k = m \), then \( F = U^*(\sum_{j=1}^m E_{ij})V \) is an extreme point of the unit ball of \( (\mathbb{F}^{m \times n}, S_\infty) \) satisfying \( (F, A) = S_1(A) \) and \( (F_1, B) = 0 \). By Proposition 2.1, condition (a) holds. If (c) holds with \( k < m \), then by a result of Thompson [10] (see also [6]), there exists \( G \in \mathbb{F}^{(m-k) \times (n-k)} \) such that \( S_\infty(G) \leq 1 \) and \( 0 = \text{tr} (B_{11}) + \text{tr} (B_{22}G^*) \). Suppose \( G = HX \) for some positive semi-definite \( H \) and \( X \in \mathbb{F}^{(m-k) \times (n-k)} \) satisfying \( XX^* = I_{m-k} \). Let \( G_1 = (H + i\sqrt{I - H^2})X, G_2 = (H - i\sqrt{I - H^2})X \), and \( F_1, F_2 \in \mathbb{F}^{m \times n} \) be such that
\[ UF_jV^* = \begin{pmatrix} I_k & 0 \\ 0 & G_j \end{pmatrix} \]
for \( j = 1, 2 \).

Then \( F_jF_j^* = I_m, (F_j, A) = S_1(A) \) and \( ((F_1 + F_2)/2, B) = 0 \). By Proposition 2.1, condition (a) holds. \( \Box \)

4 Operator norms, and a problem of Bhatia and Šemrl

Let \( \nu \) be a norm on \( \mathbb{F}^n \), and let \( \cdot \| \nu \) be the operator norm on \( \mathbb{F}^{n \times n} \) induced by \( \nu \), i.e.,
\[ \|A\|_\nu = \max\{|\nu(Ax) : x \in \mathbb{F}^n, \nu(x) \leq 1\}. \]
The dual norm of \( \nu \) and \( \cdot \| \nu \) are defined as
\[ \nu^D(x) = \max\{|\langle x, y \rangle : y \in \mathbb{F}^n, \nu(y) \leq 1\}, \]
and
\[ \|A\|_\nu^D = \max\{|\langle A, B \rangle : B \in \mathbb{F}^{n \times n}, \|B\|_\nu \leq 1\}, \]
respectively. We have the following result.

**Proposition 4.1** Let \( \nu \) be a norm on \( \mathbb{F}^n \). Denote by \( \mathcal{E} \) and \( \mathcal{E}^D \) the set of extreme points of the unit norm balls of \( \nu \) and \( \nu^D \), respectively. Then \( A \) is an extreme point of the unit ball of \( \| \cdot \|_\nu^D \) if and only if \( A = xy^* \) such that \( x \in \mathcal{E}^D \) and \( y \in \mathcal{E} \).

**Proof.** Let \( \mathcal{E}_{\| \cdot \|_\nu^D} \) be the set of extreme points of the unit ball \( \mathcal{B}_{\| \cdot \|_\nu^D} \) of \( \| \cdot \|_\nu^D \). Since
\[ \|A\|_\nu = \max\{|\langle A, X \rangle : X \in \mathbb{F}^{n \times n}, \|X\|^D \leq 1\}, \]
and
\[ \|A\|_\nu^D = \max\{|\langle A, X \rangle : X \in \mathbb{F}^{n \times n}, \|X\|^D \leq 1\}, \]
then
\[ \|A\|_\nu^D = \max\{|\langle A, X \rangle : X \in \mathbb{F}^{n \times n}, \|X\|^D \leq 1\} \]
we see that \[ \mathcal{E}_{\|\cdot\|_D} \subseteq \{xy^* : \nu^D(x) = \nu(y) = 1\}. \]

If \( y = (y_1 + y_2)/2 \), then \( xy^* = (xy_1^* + xy_2^*)/2 \); if \( x = (x_1 + x_2)/2 \), then \( xy^* = (x_1y_1^* + x_2y_2^*)/2 \). Thus,

\[ \mathcal{E}_{\|\cdot\|_D} \subseteq \{xy^* : x \in \mathcal{E}^D, \ y \in \mathcal{E}\}. \]

Suppose \( xy^* \) with \( x \in \mathcal{E}^D \) and \( y \in \mathcal{E} \). If \( xy^* \) is not an extreme point of \( \mathcal{B}_{\|\cdot\|_D} \), then it is a convex combination of other matrices in \( \{uv^* : u \in \mathcal{E}^D, \ v \in \mathcal{E}\} \), say,

\[ xy^* = \sum_{j=1}^{m} t_j x_j y_j^*, \ t_1, \ldots, t_m > 0, \ t_1 + \cdots + t_m = 1. \]

Let \( u \in \mathcal{E}^D \) be such that \( y^* u = 1 \), and let \( y_j^* u = \mu_j \) for \( j = 1, \ldots, n \). Then \( |\mu_j| \leq 1 \) and

\[ x = xy^* u = \sum_{j=1}^{t} t_j x_j y_j^* u = \sum_{j=1}^{t} t_j \mu_j x_j. \]

Since \( x \in \mathcal{E}^D \), it follows that \( \mu_j x_j = x \) with \( |\mu_j| = 1 \) for all \( j = 1, \ldots, m \). By a similar argument, we see that \( \eta_j y_j = y \) for some \( \eta_j \) with \( |\eta_j| = 1 \) for all \( j = 1, \ldots, m \). Hence

\[ xy^* = xy^* \left( \sum_{j=1}^{m} t_j \mu_j \eta_j \right). \]

Thus, \( \mu_j \eta_j = 1 \) and \( x_j y_j^* = xy^* \), which is a contradiction. So, \( \mathcal{E}_{\|\cdot\|_D} = \{xy^* : x \in \mathcal{E}^D, \ y \in \mathcal{E}\} \) as asserted. \( \square \)

Using the above result and Proposition 2.1, one readily deduces the following.

**Proposition 4.2** Suppose \( \| \cdot \|_\nu \) is an operator norm on \( \mathbb{F}^{n \times n} \) induced by the vector norm \( \nu \) on \( \mathbb{F}^n \). Given \( A \in \mathbb{F}^{n \times n} \), let

\[ V(A) = \{xy^* : x \in \mathcal{E}^D, \ y \in \mathcal{E}, \ (A, xy^*) = \|A\|_\nu \}. \]

Then \( B \in \mathbb{F}^{n \times n} \) satisfies

\[ \|A + \mu B\|_\nu \geq \|A\|_\nu \quad \text{for all} \ \mu \in \mathbb{F} \]

if and only if there exist extreme points \( x_1 y_1^*, \ldots, x_h y_h^* \in V(A) \) with \( h \leq 3 \) in the complex case and \( h \leq 2 \) in the real case, and positive numbers \( t_1, \ldots, t_h \) with \( t_1 + \cdots + t_h = 1 \) such that

\[ \sum_{j=1}^{h} t_j(B, x_j y_j^*) = 0. \tag{2} \]

Suppose \( \nu \) is a norm on \( \mathbb{F}^n \) and \( \| \cdot \|_\nu \) is the corresponding operator norm on \( \mathbb{F}^{n \times n} \). Consider the following conditions for \( A, B \in \mathbb{F}^{n \times n} \).
(I) \( \|A + \mu B\|_\nu \geq \|A\|_\nu \) for all \( \mu \in F \).

(II) There exists a vector \( y \in F^n \) with \( \nu(y) = 1 \) such that \( \nu(Ay) = \|A\|_\nu \) and

\[
\nu(Ay + \mu By) \geq \nu(Ay) \quad \text{for all } \mu \in F.
\]

In general, we have (II) implies (I). In [1], the authors conjectured that (I) also implies (II). We use Proposition 4.2 to show that this is not true in general.

**Example 4.3** Let \( \| \cdot \| \) be the operator norm on \( F^{n \times n} \) induced by the \( l_p \) norm with \( p \neq 2 \). Consider \( A = A_1 \oplus 0_{n-2} \) and \( B = I_2 \oplus 0_{n-2} \), where

\[
A_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.
\]

Define \( V(A) \) as in the proof of Proposition 4.2. Clearly, if \( Z \in V(A) \), then \( Z = Z_i \oplus 0_{n-2} \) for some \( Z_i = (a_1, a_2)^t(\overline{b}_1, \overline{b}_2) \) such that

\[
(A_1, Z_i) \geq |(A_1, uv^*)|
\]

for any \( u, v \in F^2 \) with \( l_p(u) = l_p(v) = 1 \), where \( 1/p + 1/q = 1 \). It follows from [8, Proposition 2] that \( Z_i \in V(A_1) \) with

\[
V(A_1) = \begin{cases} 
\{ \gamma(1, 1)^t(1, 0), \gamma(1, -1)^t(0, 1) \} & \text{if } p < 2 \\
\{ \gamma(1, 0)^t(1, 1), \gamma(0, 1)^t(1, -1) \} & \text{if } p > 2
\end{cases}
\]

where \( \gamma = 1/l_r((1, 1)^t) \) with \( r = \max\{p, q\} \). Let \( V(A) = \{U_1 \oplus 0_{n-2}, U_2 \oplus 0_{n-2}\} \), where \( V(A_1) = \{U_1, U_2\} \), and let \( x_j, y_j \in F^n \) satisfy \( l_p(y_j) = l_q(x_j) = 1 \) and \( x_j y_j^* = U_j \oplus 0_{n-2} \) for \( j = 1, 2 \). Then (2) holds with \( t_1 = t_2 = 1/2 \), and hence condition (I) follows.

Now, if \( y \in F^n \) satisfies \( l_p(Ay) = \|A\| \), then \( y = (b_1, b_2, 0, \ldots, 0)^t \in F^n \) and

\[
l_p(Ay) = l_p(A_1(b_1, b_2)^t).
\]

By [8, Proposition 2], we see that

(i) \( p < 2 \) and \( (b_1, b_2) \) is a multiple of \( (1, 0) \) or \( (0, 1) \), or

(ii) \( p > 2 \) and \( (b_1, b_2) \) is a multiple of \( (1, 1) \) or \( (1, -1) \).

In any case, it is impossible to have

\[
l_p((A_1 + \mu B_1)(b_1, b_2)^t) \geq l_p(A_1(b_1, b_2)^t) \quad \text{for all } \mu \in F,
\]

and thus (II) cannot hold.
References


