Extension of the Total Least Square Problem Using General Unitarily Invariant Norms [1]

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Abstract

Let m, n, p be positive integers such that $m \geq n + p$. Suppose $(A, B) \in \mathbf{C}^{m\times n} \times \mathbf{C}^{m\times p}$, and let

$$\mathcal{P}(A, B) = \{(E, F) \in \mathbf{C}^{m\times n} \times \mathbf{C}^{m\times p} : \text{there is } X \in \mathbf{C}^{n\times p} \text{ such that } (A - E)X = B - F\}.$$

The total least square problem concerns the determination of the existence of (E, F) in $\mathcal{P}(A, B)$ having the smallest Frobenius norm. In this paper, we characterize elements of the set $\mathcal{P}(A, B)$ and derive a formula for

$$\rho(A, B) = \inf \{\|[E|F]\| : (E, F) \in \mathcal{P}(A, B)\},$$

for any unitarily invariant norm $\|\cdot\|$ on $\mathbf{C}^{m\times(n+p)}$, where $[E|F]$ denotes the $m\times(n+p)$ matrix formed by the columns of E and F. Furthermore, we give a necessary and sufficient condition on (A, B) and the unitarily invariant norm $\|\cdot\|$ so that there exists $(E, F) \in \mathcal{P}(A, B)$ attaining $\rho(A, B)$. The results cover those on the total least square problem, and those of Huang and Yan on the existence of $(E, F) \in \mathcal{P}(A, B)$ so that $[E|F]$ has the smallest spectral norm.

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1 Introduction

Let $A \in \mathbf{C}^{m\times n}$ and $b \in \mathbf{C}^m$. The classical *least square problem* concerns the determination of $x \in \mathbf{C}^n$ such that the vector $f = b - Ax$ has the smallest ℓ_2 norm. In other words, one wants to determine the vectors f in the set $\mathcal{P} = \{g \in \mathbf{C}^m : Ax = b - g \text{ is solvable}\}$ with the smallest ℓ_2 norm. It is well known that if $b = b_0 + f$, where b_0 belongs to the column space V of A and f belongs to the orthogonal complement of V, then f is the vector in $\mathcal{P}$ having the minimum ℓ_2 norm.

More generally, one may consider the set $\mathcal{P}(A, b)$ of all $(E, f) \in \mathbf{C}^{m\times n} \times \mathbf{C}^m$ so that the modified linear system

$$(A - E)x = b - f$$

is solvable, and one would like to construct $(E, f) \in \mathcal{P}(A, b)$ with the smallest Frobenius norm $\|[E|f]\|_{Fr} = \{\mathrm{tr}\,(E^*E + f^*f)\}^{1/2}$, where $[E|f]$ denotes the $m \times (n+1)$ matrix formed by the columns of E and f. This is known as the *total least square problem.* Clearly, if $E = 0$ and f is the least square solution, then $(E, f) \in \mathcal{P}(A, B)$. Thus, the total least square solution (E, f) often has a smaller Frobenius norm comparing with the least square solution. However, in general, it is not easy to determine the smallest norm for those pairs $(E, f) \in \mathcal{P}(A, b)$, and it is sometimes impossible to construct (E, f) attaining the smallest Frobenius norm value. Here is an example.

Example 1.1 Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then for any $d > 0$ and

$$E_\varepsilon = \begin{bmatrix} 0 & 0 \\ 0 & \varepsilon \end{bmatrix} \quad \text{and} \quad f_\varepsilon = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

we have $(E_\varepsilon, f_\varepsilon) \in \mathcal{P}(A, b)$. So,

$$\inf_{(E,f)\in\mathcal{P}(A,b)} \|(E, f)\|_{Fr} = 0.$$

However, $\|(E, f)\|_{Fr} = 0$ if and only if $(E, f) = (0, 0)$. Evidently, $(0, 0) \notin \mathcal{P}(A, b)$. Thus, there is no element in $\mathcal{P}(A, b)$ attaining the value 0.

Many researchers have studied the total least square problem and its extension to the matrix equation

$$AX = B$$

for $A \in \mathbf{C}^{m\times n}$ and $B \in \mathbf{C}^{m\times p}$; see [1, 5, 6, 8, 10, 11]. In particular, conditions for the existence of elements (E, F) in the set

$$\mathcal{P}(A, B) = \{(E, F) \in \mathbf{C}^{m\times n} \times \mathbf{C}^{m\times p} : \text{there is } X \in \mathbf{C}^{n\times p} \text{ such that } (A - E)X = B - F\}$$

attaining the smallest Frobenius norm $\|[E|F]\|_{Fr} = \{\mathrm{tr}\,(E^*E + F^*F)\}^{1/2}$ are determined, where $[E|F]$ denotes the $m \times (n + p)$ matrix formed by the columns of E and F.

Theorem 1.2 *Let $A \in \mathbf{C}^{m\times n}$ and $B \in \mathbf{C}^{m\times p}$. Suppose $W \in \mathbf{C}^{(n+p)\times(n+p)}$ is unitary such that*

$$W^*[A|B]^*[A|B]W = \operatorname{diag}(s_1^2, \ldots, s_{n+p}^2) \quad \text{with } s_1 \geq \cdots \geq s_{n+p} \geq 0.$$

Assume $b \leq n < d \leq n+p$ are such that $s_b > s_{b+1} = \cdots = s_d > s_{d+1}$, where $s_{d+1} = 0$ if $d = n+p$, and

$$W = \begin{array}{c} n \\ p \end{array} \begin{bmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \end{bmatrix}, \atop \quad\quad b \quad\quad d-b \quad n+p-d\,.$$

Then

$$\inf\{\|[E|F]\|_{Fr} : (E,F) \in \mathcal{P}(A,B)\} = \left\{\sum_{j=1}^{p} s_{n+j}^2\right\}^{1/2},$$

and there exists (E,F) in $\mathcal{P}(A,B)$ attaining the infimum if and only if rank W_{23} has rank $n+p-d$ and $[W_{22}|W_{23}]$ has rank p.

In [3], the authors determined the condition for the existence of $(E,F) \in \mathcal{P}(A,B)$ such that $[E|F]$ attains the smallest spectral norm on $\mathbf{C}^{m\times(n+p)}$ defined by

$$\|X\|_{Sp} = \max\{(v^*X^*Xv)^{1/2} : v \in \mathbf{C}^{n+p},\ v^*v = 1\}.$$

Theorem 1.3 *Let $A \in \mathbf{C}^{m\times n}$, $B \in \mathbf{C}^{m\times p}$, $W \in \mathbf{C}^{(n+p)\times(n+p)}$, nonnegative numbers $s_1 \geq \cdots \geq s_{n+p}$, and positive integers b, d satisfy the hypotheses of Theorem 1.2. Then*

$$\inf\{\|[E|F]\|_{Sp} : (E,F) \in \mathcal{P}(A,B)\} = s_{n+1},$$

and there exists (E,F) in $\mathcal{P}(A,B)$ attaining the infimum if and only if W_{22} has rank at least $d-n$.

The Frobenius norm and the spectral norm are special instances of *unitarily invariant norms*, i.e., norms $\|\cdot\|$ that satisfy $\|UXV\| = \|X\|$ for all $X \in \mathbf{C}^{m\times(n+p)}$ and unitary matrices $U \in \mathbf{C}^{m\times m}$ and $V \in \mathbf{C}^{(n+p)\times(n+p)}$; see [7, 9] and their references for general background of unitarily invariant norms. It is interesting that in both Theorems 1.2 and 1.3, the smallest norm value of $(E,F) \in \mathcal{P}(A,B)$ is expressed in terms of the singular values of $[A|B]$, and the existence of $(E,F) \in \mathcal{P}(A,B)$ attaining the smallest norm value is determined by the ranks of certain submatrices of a unitary matrix W such that $W^*[A|B]^*[A|B]W$ is in diagonal form. In this paper, we show that the same is actually true for any unitarily invariant norm on $\mathbf{C}^{m\times(n+p)}$. In Section 2, we characterize the elements in the set $\mathcal{P}(A,B)$ and determine the value $\rho(A,B)$ for an arbitrary unitarily invariant norm $\|\cdot\|$. We then use the results to determine the condition for the existence of $(E,F) \in \mathcal{P}(A,B)$ attaining $\rho(A,B)$ in Section 3.

In our discussion, $\{E_{11}, E_{12}, \ldots, E_{m,n+p}\}$ denotes the standard basis for $\mathbf{C}^{m\times(n+p)}$. We always assume that $m \geq n+p$; otherwise, we may append zero rows to A and B. For $X \in \mathbf{C}^{k\times\ell}$ with $k \geq \ell$, let $s(X) = (s_1(X), \ldots, s_\ell(X))$ be the vector of singular values of X such that $s_1(X) \geq \cdots \geq s_\ell(X)$.

2 Elements in $\mathcal{P}(A,B)$ and a formula for $\rho(A,B)$

We use an idea in [3] to characterize elements in $\mathcal{P}(A,B)$ in the following proposition.

Proposition 2.1 *Let $(A,B)\in \mathbf{C}^{m\times n}\times \mathbf{C}^{m\times p}$ be given. Then $(E,F)\in\mathcal{P}(A,B)$ if and only if any one of the following holds.*

(a) *There is an $n\times p$ matrix X such that $[E|F]\begin{bmatrix}-X\\ I_p\end{bmatrix}=[A|B]\begin{bmatrix}-X\\ I_p\end{bmatrix}$.*

(b) *There is an $(n+p)\times p$ matrix $Y=\begin{bmatrix}Y_1\\ Y_2\end{bmatrix}$ such that $Y_2\in\mathbf{C}^{p\times p}$ is invertible, $Y^*Y=I_p$, and $[E|F]Y=[A|B]Y$.*

Proof. If $(E,F)\in\mathcal{P}(A,B)$, then there is X such that $(A-E)X=B-F$. Thus, $-AX+B=-EX+F$, and condition (a) follows.

If (a) holds, then (b) holds with $Y=\begin{bmatrix}-X\\ I_p\end{bmatrix}(I_p+X^*X)^{-1/2}$.

If (b) holds, let $X=-Y_1Y_2^{-1}$. Then $-AX+B=-EX+F$, i.e., $(A-E)X=B-F$. So, $(E,F)\in\mathcal{P}(A,B)$. □

Next, we derive a formula for $\rho(A,B)$. We will use the fact that

$$\mathcal{P}(A,B)=\{(E,F)\in\mathbf{C}^{m\times n}\times\mathbf{C}^{m\times p}:\operatorname{rank}([A-E|B-F])=\operatorname{rank}(A-E)\}.$$

Theorem 2.2 *Let $\|\cdot\|$ be a unitarily invariant norm on $\mathbf{C}^{m\times(n+p)}$, and let $(A,B)\in\mathbf{C}^{m\times n}\times\mathbf{C}^{m\times p}$. If $[A|B]\in\mathbf{C}^{m\times(n+p)}$ has singular values $s_1\ge\cdots\ge s_{n+p}$, then*

$$\rho(A,B)=\left\|\sum_{j=n+1}^{n+p}s_jE_{jj}\right\|.$$

Proof. We consider two cases. First, suppose $\operatorname{rank}([A|B])\le n$. Let P be an $n\times n$ permutation matrix such that the first r columns of AP form a basis for the range space of A. Furthermore, let $\tilde{B}$ consist of t columns of B such that the first r columns of AP and the columns of $\tilde{B}$ combined form a basis for the column space of $[A|B]$. Let $E\in\mathbf{C}^{m\times n}$ such that $EP=[O_{m\times(n-t)}|\tilde{B}]$ and $F=O_{m\times p}$. Then for any $\delta>0$, the columns of the matrix $(A-\delta E)P$ span the column space of $[A|B]$. So, for any $\varepsilon>0$ there is $\delta>0$ such that $\operatorname{rank}([A-\delta E|B-F])=\operatorname{rank}(A-\delta E)$ and $\|[\delta E|F]\|<\varepsilon$. Hence,

$$\rho(A,B)=0=\left\|\sum_{j=n+1}^{n+p}s_jE_{jj}\right\|.$$

Now, suppose $\operatorname{rank}([A|B])>n$. If $(E,F)\in\mathcal{P}(A,B)$, then

$$\operatorname{rank}([A|B]-[E|F])=\operatorname{rank}(A-E)\le n.$$

By the result of Mirsky [9],

$$\|[E|F]\| \geq \left\| \sum_{j=n+1}^{n+p} s_j E_{jj} \right\|.$$

Suppose $U \in \mathbf{C}^{m\times(n+p)}$ and $V \in \mathbf{C}^{(n+p)\times(n+p)}$ are such that $U^*U = V^*V = I_{n+p}$ and $[A|B] = U\mathrm{diag}\,(s_1, \ldots, s_{n+p})V$. The matrix

$$[\tilde{A}|\tilde{B}] = U^*\mathrm{diag}\,(s_1, \ldots, s_n, 0, \ldots, 0)V$$

has rank n. By the proof in the preceding paragraph, there is $\tilde{E} \in \mathbf{C}^{m\times n}$ such that

$$\mathrm{rank}\,([\tilde{A} - \delta\tilde{E}|\tilde{B}]) = \mathrm{rank}\,(\tilde{A} - \delta\tilde{E})$$

for any $\delta > 0$. Thus for any $\varepsilon > 0$, we can construct $(E, F) \in \mathcal{P}(A, B)$ such that

$$[E|F] = U(\mathrm{diag}\,(0, \ldots, 0, s_{n+1}, \ldots, s_{n+p}))V + [\delta\tilde{E}|O_{n\times p}]$$

and

$$\|[E|F]\| < \left\| \sum_{j=n+1}^{n+p} s_j E_{jj} \right\| + \varepsilon.$$

Hence, $\rho(A, B) = \| \sum_{j=n+1}^{n+p} s_j E_{jj}\|$ as asserted. □

3 Existence of elements in $\mathcal{P}(A, B)$ attaining $\rho(A, B)$

Let $\|\cdot\|$ be a norm on $\mathbf{C}^{m\times(n+p)}$, and let $(A, B) \in \mathbf{C}^{m\times n} \times \mathbf{C}^{m\times p}$. We say that $\rho(A, B)$ is *attainable* if there is $(E, F) \in \mathcal{P}(A, B)$ such that $\|[E|F]\| = \rho(A, B)$.

Proposition 3.1 *Let $\|\cdot\|$ be a norm on $\mathbf{C}^{m\times(n+p)}$, and let $(A, B) \in \mathbf{C}^{m\times n} \times \mathbf{C}^{m\times p}$. Then $\rho(A, B) = 0$ is attainable (by $(E, F) = (O, O)$) if and only if* $\mathrm{rank}\,([A|B]) = \mathrm{rank}\,(A)$. *In case $\|\cdot\|$ is unitarily invariant, $\rho(A, B) = 0$ if and only if* $\mathrm{rank}\,([A|B]) \leq n$.

Proof. The first assertion can be verified readily. The second assertion follows from Theorem 2.2. □

The problem is more delicate if $\rho(A, B) > 0$. We need some more notation and facts about unitarily invariant norms. Given two real vectors $x = (x_1, \ldots, x_\ell)$ and $y = (y_1, \ldots, y_\ell)$ in $\mathbf{R}^{1\times\ell}$ with $\ell \leq n + p$, we say that y is *weakly majorized* by x, denoted by $y \prec_w x$, if the sum of the t largest entries of y is not larger than that of x for $t = 1, \ldots, \ell$. In addition, if all the entries of x and y are nonnegative, then for any unitarily invariant norm $\|\cdot\|$ on $\mathbf{C}^{m\times(n+p)}$, we have

$$\left\| \sum_{j=1}^{p} y_j E_{jj} \right\| \leq \left\| \sum_{j=1}^{p} x_j E_{jj} \right\|;$$

see [9].

Theorem 3.2 *Let $\|\cdot\|$ be a unitarily invariant norm on $\mathbf{C}^{m\times(n+p)}$, and let $(A,B)\in\mathbf{C}^{m\times n}\times\mathbf{C}^{m\times p}$. The following conditions are equivalent.*

(a) *There is $(E,F)\in\mathcal{P}(A,B)$ such that $\|[E|F]\| = \rho(A,B)$.*

(b) *There is an $(n+p)\times p$ matrix $Y=\begin{bmatrix}Y_1\\Y_2\end{bmatrix}$ such that $Y_2\in\mathbf{C}^{p\times p}$ is invertible, $Y^*Y=I_p$, and for $[Y|0]\in\mathbf{C}^{(n+p)\times(n+p)}$ we have*

$$\|[A|B][Y|0]\| = \left\|\sum_{j=1}^{p} s_j([A|B]Y)E_{jj}\right\| = \left\|\sum_{j=n+1}^{n+p} s_j([A|B])E_{jj}\right\|.$$

Proof. Suppose (b) holds. Let $[E|F]=[A|B]YY^*$. Then $[E|F]Y=[A|B]YY^*Y=[A|B]Y$, and hence $(E,F)\in\mathcal{P}(A,B)$ by Proposition 2.1. Let U be a unitary matrix of the form $[Y|Z]$. Then

$$\|[E|F]\| = \|[E|F]U\| = \|[A|B]YY^*[Y|Z]\| = \|[A|B][Y|0]\| = \rho(A,B).$$

Conversely, suppose there exists $(E,F)\in\mathcal{P}(A,B)$ such that $\|[E|F]\|=\rho(A,B)$. By Proposition 2.1, there is Y satisfying Proposition 2.1 (b) such that $[E|F]Y=[A|B]Y$. So, there exists a unitary $U\in\mathbf{C}^{(n+p)\times(n+p)}$ of the form $[Y|Z]$ such that $[E|F]Y=[A|B]Y$ is a submatrix of $[A|B]U$. By the result in [12], we have

$$s_j([A|B][Y|0]) \ge s_{n+j}([A|B]) \qquad \text{for } j=1,\dots,p.$$

Thus, $\|[A|B]\|\ge\|[A|B][Y|0]\|$. Similarly, $\|[E|F]\|\ge\|[E|F][Y|0]\|$. Hence,

$$\begin{aligned}\rho(A,B) &= \|[E|F]\| \ge \|[E|F][Y|0]\| = \|[A|B][Y|0]\| \\ &= \left\|\sum_{j=1}^{p} s_j([A|B][Y|0])E_{jj}\right\| \ge \left\|\sum_{j=n+1}^{n+p} s_j([A|B])E_{jj}\right\| = \rho(A,B).\end{aligned}$$ □

Condition (b) in Theorem 3.2 is not easy to check. We obtain a better condition, which is computable and easier to check, in Theorem 3.4. We first prove an auxiliary lemma.

Lemma 3.3 *Suppose a unitarily invariant norm $\|\cdot\|$ on $\mathbf{C}^{m\times(n+p)}$ and $X\in\mathbf{C}^{m\times(n+p)}$ are given so that*

$$\|X\| = \left\|\sum_{j=1}^{r} s_j(X)E_{jj}\right\| > \left\|\sum_{j=1}^{r-1} s_j(X)E_{jj}\right\|.$$

Assume that $s_r(X)=\cdots=s_t(X)>s_{t+1}(X)$ for some positive integer $t\ge r$, and $Z\in\mathbf{C}^{m\times(n+p)}$ such that $s_j(Z)\ge s_j(X)$ for all $j=1,\dots,n+p$. If $\|X\|=\|Z\|$, then $s_j(X)=s_j(Z)$ for all $j=1,\dots,t$.

Proof. Let $\|\cdot\|$ and X satisfy the hypotheses of the lemma. Then (see [4, 7]), there exists a compact subset $\mathcal{K}$ of nonnegative vectors of the form $(c_1, \ldots, c_{n+p})$ with $c_1 \geq \cdots \geq c_{n+p}$ such that

$$\|Y\| = \max\{\|Y\|_c : c \in \mathcal{K}\},$$

where

$$\|Y\|_c = \sum_{j=1}^{n+p} c_j s_j(Y).$$

By the assumption on X and the norm $\|\cdot\|$, we see that $s_r(X) > 0$ and there is a vector $c = (c_1, \ldots, c_{n+p}) \in \mathcal{K}$ with $c_r > 0$ such that

$$\|X\| = \|X\|_c.$$

Assume that there is $j \in \{1, \ldots, t\}$ such that $s_j(Z) > s_j(X)$. If $j > r$, then $s_r(Z) \geq s_j(Z) > s_j(X) = s_r(X)$. So, we may assume that $j \leq r$. Thus, $\sum_{j=1}^{r} c_j(s_j(Z) - s_j(X)) > 0$, and

$$\|Z\| \geq \|Z\|_c > \|X\|_c = \|Z\|,$$

which is a contradiction. □

Theorem 3.4 *Let $\|\cdot\|$ be a unitarily invariant norm on $\mathbf{C}^{m\times(n+p)}$, and let $(A, B) \in \mathbf{C}^{m\times n} \times \mathbf{C}^{m\times p}$ be such that $[A|B]$ has singular values $s_1 \geq \cdots \geq s_{n+p}$ with $s_n > 0$. Suppose*

$$\rho(A, B) = \left\|\sum_{j=1}^{p} s_{n+j} E_{jj}\right\| = \left\|\sum_{j=1}^{r} s_{n+j} E_{jj}\right\| > \left\|\sum_{j=1}^{r-1} s_{n+j} E_{jj}\right\|.$$

Let $b \leq n < d \leq n+t$ satisfy $s_b > s_{b+1} = \cdots = s_d > s_{d+1}$ and $s_{n+r} = \cdots = s_{n+t} > s_{n+t+1}$,

$$W = \begin{array}{c} n \\ p \end{array}\!\! \begin{bmatrix} W_{11} & W_{12} & W_{13} & W_{14} \\ W_{21} & W_{22} & W_{23} & W_{24} \end{bmatrix}, \quad \begin{array}{cccc} b & d-b & n+t-d & p-t \end{array}$$

be unitary such that $W^*[A|B]^*[A|B]W = \operatorname{diag}(s_1^2, \ldots, s_{n+p}^2)$. The following conditions are equivalent.

(a) *There is $(E, F) \in \mathcal{P}(A, B)$ such that $\|[E|F]\| = \rho(A, B)$.*

(b) *W_{23} has rank $n + t - d$ and $[W_{22}|W_{23}]$ has rank at least t.*

(c) *There is an orthonormal family $\{v_1, \ldots, v_t\}$ of eigenvectors for the matrix $[A|B]^*[A|B]$ corresponding to the eigenvalues $s_{n+1}^2, \ldots, s_{n+t}^2$ such that $[v_1|\cdots|v_t] = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$ for a $p \times t$ V_2 matrix having rank t.*

Proof. It is easy to verify (b) $\iff$ (c). Suppose (c) holds. Let V be the $(n+p)\times t$ matrix with $v_1,\ldots,v_t$ as columns. One can find an orthonormal set of vectors $y_{t+1},\ldots,y_p$ such that $V^*y_j=0$ and $\|Ay_j\|\le s_{n+t}$ for $j=t+1,\ldots,p$. Let $Y=[V|y_{t+1}|\cdots|y_p]$. Then Theorem 3.2 (a) holds, and hence condition (a) holds by Theorem 3.2 .

Suppose (a) holds. Then Theorem 3.2 (b) holds. Let Y be the matrix satisfying Theorem 3.2 (b). If $[A|B]Y$ has singular values $\mu_1\ge\cdots\ge\mu_p$, then $\mu_j\ge s_{n+j}$ for $j=1,\ldots,p$, by the result in [12]. Since

$$\left\|\sum_{j=1}^{p} s_{n+j}E_{jj}\right\| = \left\|\sum_{j=1}^{r} s_{n+j}E_{jj}\right\|,$$

we see that $\mu_j=s_{n+j}$ for $j=1,\ldots,t$ by Lemma 3.3. Suppose U is a unitary matrix of the form $[Y|Z]$. Then

$$U^*[A|B]^*[A|B]U=\begin{bmatrix} P & R^* \\ R & Q\end{bmatrix}$$

such that P is $p\times p$ and has eigenvalues $\mu_1^2,\ldots,\mu_p^2$. Since $\mu_j=s_{n+j}$ for $j=1,\ldots,t$, we can apply [2, Theorem 2.1 (ii)] to conclude that the column spaces of Y contains an orthonormal set $\mathcal{V}$ with at least t elements from the subspace spanned by the eigenvectors of the matrix $[A|B]^*[A|B]$ corresponding to the eigenvalues $s_{b+1}^2,\ldots,s_{n+t}^2$. Since $\mu_j=s_{n+j}$ for $j=1,\ldots,d-n$, we see that there are at least $d-n$ eigenvectors in $\mathcal{V}$ corresponding to the eigenvalue s_{n+1}^2. Since $\mu_{d-n+1}=s_{d+1}<s_d$, we see that there are at most $d-n$ eigenvectors in $\mathcal{V}$ corresponding to the eigenvalue s_{n+1}^2. Consequently, the remaining $t-(d-n)$ eigenvectors must correspond to the eigenvalues $s_{d+1}^2,\ldots,s_{n+t}^2$. Thus, the column space of Y contains an orthonormal family $\{v_1,\ldots,v_t\}$ satisfying condition (c). □

Specializing Theorem 3.4 to the Frobenius norm, we see that the equivalence of (a) and (b) reduces to the result in [8]; see also Theorem 1.2. Note that in this case, W_{14} and W_{24} are vacuous. Moreover, if $(E,F)\in\mathcal{P}(A,B)$ satisfies $\|[E|F]\|_{Fr}=\rho(A,B)$, then $s([E|F])=(s_{n+1},\ldots,s_{n+p},0,\ldots,0)$. Hence, $\rho(A,B)$ is attained by (E,F) for any other unitarily invariant norm $\|\cdot\|$ on $\mathbf{C}^{m\times(n+p)}$. In fact, the same result holds whenever $p=t$ in the hypotheses of Theorem 3.4. We have the following.

Corollary 3.5 *Use the notation of Theorem 3.4. Suppose $t=p$. If $(E,F)\in\mathcal{P}(A,B)$ satisfies $\|[E|F]\|=\rho(A,B)$, then $\rho(A,B)$ is attained by (E,F) for any other unitarily invariant norm on $\mathbf{C}^{m\times(n+p)}$.*

We note that the hypothesis of the above corollary is satisfied by many unitarily invariant norms. For example, the Schatten q-norms defined by $S_q(X)=\{\sum_{j=1}^p s_j(X)^q\}^{1/q}$ for $q\ge 1$.

Specializing Theorem 3.4 to the spectral norm, we see that the equivalence of (a) and (b) reduces to the result in [3]; see also Theorem 1.3. Note that in this case, W_{13} and W_{23} are vacuous. Moreover, we have the following.

Corollary 3.6 *If $(E,F)\in\mathcal{P}(A,B)$ satisfies $\|[E|F]\|=\rho(A,B)$ for a given unitarily invariant norm $\|\cdot\|$, then (E,F) also attains $\rho(A,B)$ for the spectral norm.*

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