# A Note on Eigenvalues of Perturbed Hermitian Matrices

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**Abstract**

Let

$$A = \begin{pmatrix} H_1 & E^* \\ E & H_2 \end{pmatrix} \quad \text{and} \quad \widetilde{A} = \begin{pmatrix} H_1 & O \\ O & H_2 \end{pmatrix}$$

be Hermitian matrices with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_k$ and $\widetilde{\lambda}_1 \geq \cdots \geq \widetilde{\lambda}_k$, respectively. Denote by $\|E\|$ the spectral norm of the matrix $E$, and $\eta$ the spectral gap between the spectra of $H_1$ and $H_2$. It is shown that

$$|\lambda_i - \widetilde{\lambda}_i| \leq \frac{2\|E\|^2}{\eta + \sqrt{\eta^2 + 4\|E\|^2}} ,$$

which improves all the existing results. Similar bounds are obtained for singular values of matrices under block perturbations.

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## 1 Introduction

Consider a partitioned Hermitian matrix

$$A = \begin{matrix} & \begin{matrix} m & n \end{matrix} \\ \begin{matrix} m \\ n \end{matrix} & \begin{pmatrix} H_1 & E^* \\ E & H_2 \end{pmatrix} \end{matrix}, \tag{1.1}$$

where $E^*$ is $E$'s complex conjugate transpose. At various situations (typically when $E$ is *small*), one is interested in knowing the impact of removing $E$ and $E^*$ on the eigenvalues

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of $A$. More specifically, one would like to obtain bounds for the differences between that eigenvalues of $A$ and those of its perturbed matrix

$$\widetilde{A} = \begin{matrix} & m & n \\ m & \multicolumn{2}{c}{} \\ n & & \end{matrix} \begin{pmatrix} H_1 & O \\ O & H_2 \end{pmatrix}. \tag{1.2}$$

Let $\lambda(X)$ be the spectrum of the square matrix $X$, and let $\|Y\|$ be the spectral norm of a matrix $Y$, i.e., the largest singular value of $Y$. There are two kinds of bounds for the eigenvalues $\lambda_1 \geq \cdots \geq \lambda_{m+n}$ and $\widetilde{\lambda}_1 \geq \cdots \geq \widetilde{\lambda}_{m+n}$ of $A$ and $\widetilde{A}$, respectively:

1. [1, 7, 8]
$$|\lambda_i - \widetilde{\lambda}_i| \leq \|E\|. \tag{1.3}$$

2. [1, 2, 3, 5, 7, 8] If the spectra of $H_1$ and $H_2$ are disjoint, then
$$|\lambda_i - \widetilde{\lambda}_i| \leq \|E\|^2/\eta, \tag{1.4}$$
where
$$\eta \stackrel{\text{def}}{=} \min_{\mu_1 \in \lambda(H_1),\, \mu_2 \in \lambda(H_2)} |\mu_1 - \mu_2|,$$
and $\lambda(H_i)$ is the spectrum of $H_i$.

The bounds of the first kind do not use information of the spectral distribution of the $H_1$ and $H_2$, which will give (much) weaker bounds when $\eta$ is not so small; while the bounds of the second kind may blow up whenever $H_1$ and $H_2$ have a common eigenvalue. Thus both kinds have their own drawbacks, and it would be advantageous to have bounds that are always no bigger than $\|E\|$, of $\mathcal{O}(\|E\|)$ as $\eta \to 0$, and at the same time behave like $\mathcal{O}(\|E\|^2/\eta)$ for not so small $\eta$. To further motivate our study, let us look at the following $2 \times 2$ example.

**Example 1** *Consider the* $2 \times 2$ *Hermitian matrix*

$$A = \begin{pmatrix} \alpha & \epsilon \\ \epsilon & \beta \end{pmatrix}. \tag{1.5}$$

Interesting cases are when $\epsilon$ is *small*, and thus $\alpha$ and $\beta$ are *approximate* eigenvalues of $A$. We shall analyze by how much the eigenvalues of $A$ differ from $\alpha$ and $\beta$. Without loss of generality, assume

$$\alpha > \beta.$$

The eigenvalues of $A$, denoted by $\lambda_\pm$, satisfy $\lambda^2 - (\alpha+\beta)\lambda + \alpha\beta - \epsilon^2 = 0$; and thus

$$\lambda_\pm = \frac{\alpha + \beta \pm \sqrt{(\alpha+\beta)^2 - 4(\alpha\beta - \epsilon^2)}}{2} = \frac{\alpha + \beta \pm \sqrt{(\alpha-\beta)^2 + 4\epsilon^2}}{2}.$$

Now

$$0 < \left\{ \begin{array}{c} \lambda_+ - \alpha \\ \beta - \lambda_- \end{array} \right\} = \frac{-(\alpha-\beta) + \sqrt{(\alpha-\beta)^2 + 4\epsilon^2}}{2} = \frac{2\epsilon^2}{(\alpha-\beta) + \sqrt{(\alpha-\beta)^2 + 4\epsilon^2}} \quad (1.6)$$

which provides a difference that enjoys the following properties:

$$\frac{2\epsilon^2}{(\alpha-\beta) + \sqrt{(\alpha-\beta)^2 + 4\epsilon^2}} \left\{ \begin{array}{ll} \leq \epsilon & \text{always}, \\ \to \epsilon & \text{as } \alpha \to \beta^+, \\ \leq \epsilon^2/(\alpha-\beta). & \end{array} \right.$$

The purpose of this note is to extend this $2 \times 2$ example and obtain bounds which improve both (1.3) and (1.4). Such results are not only of theoretical interest but also important in the computations of eigenvalues of Hermitian matrices [4, 6, 9].

As an application, similar bounds are presented for the singular value problem.

## 2 Main Result

**Theorem 2** *Let*

$$A = \begin{array}{c} \\ m \\ n \end{array} \!\!\begin{array}{c} \begin{array}{cc} m & n \end{array} \\ \begin{pmatrix} H_1 & E^* \\ E & H_2 \end{pmatrix} \end{array} \quad \textit{and} \quad \widetilde{A} = \begin{array}{c} \\ m \\ n \end{array} \!\!\begin{array}{c} \begin{array}{cc} m & n \end{array} \\ \begin{pmatrix} H_1 & O \\ O & H_2 \end{pmatrix} \end{array}$$

*be Hermitian matrices with eigenvalues*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{m+n} \quad \textit{and} \quad \widetilde{\lambda}_1 \geq \widetilde{\lambda}_2 \geq \cdots \geq \widetilde{\lambda}_{m+n}, \quad (2.1)$$

*respectively. Define*

$$\eta_i \stackrel{\text{def}}{=} \left\{ \begin{array}{ll} \min_{\mu_2 \in \lambda(H_2)} |\widetilde{\lambda}_i - \mu_2|, & \textit{if } \widetilde{\lambda}_i \in \lambda(H_1), \\ \min_{\mu_1 \in \lambda(H_1)} |\widetilde{\lambda}_i - \mu_1|, & \textit{if } \widetilde{\lambda}_i \in \lambda(H_2), \end{array} \right. \quad (2.2)$$

$$\eta \stackrel{\text{def}}{=} \min_{1 \leq i \leq m+n} \eta_i = \min_{\mu_1 \in \lambda(H_1),\, \mu_2 \in \lambda(H_2)} |\mu_1 - \mu_2|. \quad (2.3)$$

*Then for $i = 1, 2, \cdots, m+n$, we have*

$$|\lambda_i - \widetilde{\lambda}_i| \leq \frac{2\|E\|^2}{\eta_i + \sqrt{\eta_i^2 + 4\|E\|^2}} \quad (2.4)$$

$$\leq \frac{2\|E\|^2}{\eta + \sqrt{\eta^2 + 4\|E\|^2}}. \quad (2.5)$$

*Proof.* Suppose $U^*H_1U$ and $V^*H_2V$ are in diagonal form with diagonal entries arranged in descending order. We may assume that $U = I_m$ and $V = I_n$. Otherwise, replace $A$ by

$$(U \oplus V)^* A (U \oplus V).$$

We may perturb the diagonal of $A$ so that all entries are distinct, and apply continuity argument for the general case.

We prove the result by induction on $m+n$. If $m+n = 2$, the result is clear (from our Example). Assume that $m+n > 2$, and the result is true for Hermitian matrices of size $m+n-1$.

First, refining an argument of Mathias [5], we show that (2.4) holds for $i = 1$. Assume that the $(1,1)$ entry of $H_1$ equals $\widetilde{\lambda}_1$. By the min-max principle [1, 7, 8], we have

$$\lambda_1 \geq e_1^* A e_1 = \widetilde{\lambda}_1,$$

where $e_1$ is the first column of the identity matrix. Let

$$X = \begin{pmatrix} I_m & 0 \\ -(H_2 - \mu_1 I_n)^{-1} E & I_n \end{pmatrix}.$$

Then

$$X^*(A - \lambda_1 I)X = \begin{pmatrix} H_1(\lambda_1) & 0 \\ 0 & H_2 - \lambda_1 I_n \end{pmatrix},$$

where

$$H_1(\lambda_1) = H_1 - \lambda_1 I_m - E^*(H_2 - \lambda_1 I_n)^{-1} E.$$

Since $A$ and $X^*AX$ have the same inertia, we see that $H_1(\lambda_1)$ has zero as the largest eigenvalue. Notice that the largest eigenvalue of $H_1 - \lambda_1 I$ is $\widetilde{\lambda}_1 - \lambda_1 \leq 0$. Thus, for $\delta_1 = |\lambda_1 - \widetilde{\lambda}_1| = \lambda - \widetilde{\lambda}_1$, we have (see [7, (10.9)])

$$\lambda_1 \leq \widetilde{\lambda}_1 + \|E\|_2^2/(\delta_1 + \eta_1),$$

and hence

$$\delta_1 \leq \|E\|^2/(\delta_1 + \eta_1).$$

Consequently,

$$\delta_1 \leq \frac{2\|E\|}{\eta_1 + \sqrt{\eta_1^2 + 4\|E\|^2}}$$

as asserted. Similarly, we can prove the result if the $(1,1)$ entry of $H_2$ equals $\widetilde{\lambda}_1$. In this case, we will apply the inertia arguments to $A$ and $YAY^*$ with

$$Y = \begin{pmatrix} I_m & 0 \\ -E(H_1 - \lambda_1 I_m)^{-1} & I_n \end{pmatrix}.$$

Applying the result of the last paragraph to $-A$, we see that (2.2) holds for $i = m+n$.

Now, suppose $1 < i < m+n$. The result trivially holds if $\lambda_i = \widetilde{\lambda}_i$. Suppose $\lambda_i \neq \widetilde{\lambda}_i$. We may assume that $\widetilde{\lambda}_i > \lambda_i$. Otherwise, replace $(A, \widetilde{A}, i)$ by $(-A, -\widetilde{A}, m+n-i+1)$. Delete the row and column of $A$ that contain the diagonal entry $\lambda_n$. Suppose the resulting matrix $\widehat{A}$ has eigenvalues $\nu_1 \geq \cdots \geq \nu_{m+n-1}$. By the interlacing inequalities [7, Section 10.1], we have

$$\lambda_i \geq \nu_i \qquad \text{and hence} \qquad \widetilde{\lambda}_i - \lambda_i \leq \widetilde{\lambda}_i - \nu i. \tag{2.6}$$

Note that $\widetilde{\lambda}_i$ is the $i$th largest diagonal entries in $\widehat{A}$. Let $\widehat{\eta}_i$ be the minimum distance between $\widetilde{\lambda}_i$ and the diagonal entries in the diagonal block $\widehat{H}_j$ in $\widehat{A}$ not containing $\widetilde{\lambda}_i$; here $j \in \{1, 2\}$. Then

$$\widehat{\eta}_i \geq \eta_i$$

because $\widehat{H}_j$ may have one fewer diagonal entries than $H_j$. Let $\widehat{E}$ be the off-diagonal block of $\widehat{A}$. Then $\|\widehat{E}\| \leq \|E\|$. Thus,

$$\begin{aligned}
|\lambda_i - \widetilde{\lambda}_i| &= \widetilde{\lambda}_i - \lambda_i && \text{because } \widetilde{\lambda}_i > \lambda_i \\
&\leq \widetilde{\lambda}_i - \nu_i && \text{by (2.6)} \\
&\leq \frac{2\|\widehat{E}\|^2}{\widehat{\eta}_i + \sqrt{\widehat{\eta}_i^2 + 4\|\widehat{E}\|^2}} && \text{by induction assumption} \\
&\leq \frac{2\|\widehat{E}\|^2}{\eta_i + \sqrt{\eta_i^2 + 4\|\widehat{E}\|^2}} && \text{because } \widehat{\eta}_i \geq \eta_i \\
&= \frac{1}{2}\sqrt{\eta_i^2 + 4\|\widehat{E}\|^2} - \eta_i && \\
&\leq \frac{1}{2}\sqrt{\eta_i^2 + 4\|E\|^2} - \eta_i && \text{because } \|\widehat{E}\| \leq \|E\| \\
&= \frac{2\|E\|^2}{\eta_i + \sqrt{\eta_i^2 + 4\|E\|^2}}
\end{aligned}$$

as asserted. ∎

# 3 Application to Singular Value Problem

In this section, we apply the result in Section 2 to study singular values of matrices. For notational convenience in connection to our discussion, we define the sequence of singular values of a complex $p \times q$ matrix $X$ by

$$\sigma(X) = (\sigma_1(X), \ldots, \sigma_k(X)),$$

where $k = \max\{p, q\}$ and $\sigma_1(X) \geq \cdots \geq \sigma_k(X)$ are the nonnegative square roots of the eigenvalues of the matrix $XX^*$ or $X^*X$ depending on which one has a larger size. Note that the nonzero eigenvalues of $XX^*$ and $X^*X$ are the same, and they give rise to the nonzero singular values of $X$ which are of importance. We have the following result concerning the nonzero singular values of perturbed matrices.

**Theorem 3** *Let*

$$B = \begin{matrix} & k & \ell \\ m & \multicolumn{2}{c}{} \\ n & & \end{matrix} \begin{pmatrix} G_1 & E_1 \\ E_2 & G_2 \end{pmatrix} \quad \text{and} \quad \widetilde{B} = \begin{matrix} m \\ n \end{matrix} \begin{pmatrix} G_1 & O \\ O & G_2 \end{pmatrix}$$

*be complex matrices with singular values*

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\max\{m+n,k+\ell\}} \quad \text{and} \quad \widetilde{\sigma}_1 \geq \widetilde{\sigma}_2 \geq \cdots \geq \widetilde{\sigma}_{\max\{m+n,k+\ell\}}, \tag{3.1}$$

*respectively, so that* $G_1$ *and* $G_2$ *are non-trivial. Define* $\epsilon = \max\{\|E_1\|, \|E_2\|\}$*, and*

$$\eta_i \stackrel{\text{def}}{=} \begin{cases} \min\limits_{\mu_2 \in \sigma(G_2)} |\widetilde{\sigma}_i - \mu_2|, & \text{if } \widetilde{\sigma}_i \in \sigma(G_1), \\ \min\limits_{\mu_1 \in \sigma(G_1)} |\widetilde{\sigma}_i - \mu_1|, & \text{if } \widetilde{\sigma}_i \in \sigma(G_2), \end{cases} \tag{3.2}$$

$$\eta \stackrel{\text{def}}{=} \min_{1 \leq i \leq m+n} \eta_i = \min_{\mu_1 \in \sigma(G_1),\, \mu_2 \in \sigma(G_2)} |\mu_1 - \mu_2|. \tag{3.3}$$

*Then for* $i = 1, 2, \cdots, \min\{m+n, k+\ell\}$*, we have*

$$|\sigma_i - \widetilde{\sigma}_i| \leq \frac{2\epsilon^2}{\eta_i + \sqrt{\eta_i^2 + 4\epsilon^2}} \tag{3.4}$$

$$\leq \frac{2\epsilon^2}{\eta + \sqrt{\eta^2 + 4\epsilon^2}}, \tag{3.5}$$

*and* $\sigma_i = \widetilde{\sigma}_i = 0$ *for* $i > \min\{m+n, k+\ell\}$*.*

Proof: By Jordan-Wielandt Theorem [8, Theorem I.4.2], the eigenvalues of

$$\begin{pmatrix} O & B \\ B^* & O \end{pmatrix}$$

are $\pm\sigma_i$ and possibly some zeros adding up to $m+n+k+\ell$ eigenvalues. A similar statement holds for $\widetilde{B}$. Permuting the rows and columns appropriately, we see that

$$\begin{pmatrix} O & B \\ B^* & O \end{pmatrix} \quad \text{is similar to} \quad \left(\begin{array}{cc|cc} O & G_1 & O & E_1 \\ G_1^* & O & E_2^* & O \\ \hline O & E_2 & O & G_2 \\ E_1^* & O & G_2^* & O \end{array}\right),$$

and

$$\begin{pmatrix} O & \widetilde{B} \\ \widetilde{B}^* & O \end{pmatrix} \quad \text{is similar to} \quad \left(\begin{array}{cc|cc} O & G_1 & & \\ G_1^* & O & & \\ \hline & & O & G_2 \\ & & G_2^* & O \end{array}\right).$$

Applying Theorem 2 with

$$H_i = \begin{pmatrix} O & G_i \\ G_i^* & O \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} O & E_2 \\ E_1^* & O \end{pmatrix},$$

we get the result. ∎

One can also apply the above proof to the degenerate cases when $G_1$ or $G_2$ in the matrix $B$ is trivial, i.e., one of the parameters $m, n, k, \ell$ is zero. These cases are useful in applications. We state one of them, and one can easily extend it to other cases.

**Theorem 4** *Suppose* $B = (\, G \;\; E \,)$ *and* $\tilde{B} = (\, G \;\; O \,)$ *are* $p \times q$ *matrices with singular values*

$$\sigma_1 \geq \cdots \geq \sigma_{\max\{p,q\}} \quad \text{and} \quad \widetilde{\sigma}_1 \geq \ldots \geq \widetilde{\sigma}_{\max\{p,q\}},$$

*respectively. Then for* $i = 1, \ldots, \min\{p, q\}$,

$$|\sigma_i - \widetilde{\sigma}_i| \leq \frac{2\|E\|}{2\widetilde{\sigma}_i + \sqrt{\widetilde{\sigma}_i^2 + 4\|E\|^2}}.$$

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