A short proof of interlacing inequalities on normalized Laplacians

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Abstract
A short proof of interlacing inequalities on normalized Laplacians is given.

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1 Introduction

Let G be simple graph with adjacency matrix $A = A(G)$ and Laplacian $L = L(G)$. Then $L = D - A$ with $D = D(G) = \mathrm{diag}\,(d_1, \dots, d_n)$ such that $d_1, \dots, d_n$ are the degrees of the vertices of G.

The normalized Laplacian of G is defined as $\mathcal{L}(G) = TLT$, where T is the diagonal matrix $\mathrm{diag}\,(t_1, \dots, t_n)$ such that $t_j = 1/\sqrt{d_j}$ if $d_j \neq 0$ and $t_j = 1$ otherwise. Normalized Laplacians have many interesting properties and are very useful in studying graphs; see [2] and its references. In this note, we give a short proof of the following interesting result obtained in [1] recently.

Theorem *Suppose H is a connected graph obtained from the graph G by removing an edge. Let $\mathcal{L}(G)$ and $\mathcal{L}(H)$ have eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \cdots \geq \mu_n$, respectively. Set $\lambda_0 = 2$ and $\lambda_{n+1} = 0$. Then $\lambda_{j-1} \geq \mu_j \geq \lambda_{j+1}$ for $j = 1, \dots, n$.*

The proof in [1, Section 3] used the Courant-Fischer theorem in the context of harmonic eigenfunctions and some intricate calculation. Ours depends on the following elementary facts and some simple 2×2 (block) matrix manipulations.

1. The eigenvalues of $\mathcal{L}(G)$ lies in $[0, 2]$. [To see this, observe that $\mathcal{L}(G)$ and $T^{-1}\mathcal{L}(G)T$ have the same eigenvalues, and each eigenvalue ξ of the latter matrix satisfies $|\xi - 1| \leq 1$ by the Gershgorin theorem.]

2. For any symmetric matrix A and unit vector v, the value v^tAv lies between the smallest and largest eigenvalues of A. [This is the Rayleigh principle.]

2 Proof of Theorem

We may relabel the vertices and assume that H is obtained from G by removing the edge joining vertex 1 and vertex 2. Let $L(G) = \begin{pmatrix} X & Y \\ Y^t & Z \end{pmatrix}$. Suppose $D_1 = \mathrm{diag}\,(1/\sqrt{d_1}, 1/\sqrt{d_2})$, $\widetilde{D}_1 = \mathrm{diag}\,(1/\sqrt{d_1 - 1}, 1/\sqrt{d_2 - 1})$, and $D_2 = \mathrm{diag}\,(1/\sqrt{d_3}, \dots, 1/\sqrt{d_n})$. Then

$$\mathcal{L}(G) = \begin{pmatrix} D_1XD_1 & D_1YD_2 \\ D_2Y^tD_1 & D_2ZD_2 \end{pmatrix} \quad \text{and} \quad \mathcal{L}(H) = \begin{pmatrix} I_2 & \widetilde{D}_1YD_2 \\ D_2Y^t\widetilde{D}_1 & D_2ZD_2 \end{pmatrix}.$$

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To get the desired conclusion, we show that for any $\mu \in (\mu_n, \mu_1)$ such that $D_2 Z D_2 - \mu I_{n-2}$ is invertible,

(a) if $\mathcal{L}(H) - \mu I_n$ has p positive eigenvalues then $\mathcal{L}(G) - \mu I_n$ has at least $p-1$ positive eigenvalues;

(b) if $\mathcal{L}(H) - \mu I_n$ has q negative eigenvalues then $\mathcal{L}(G) - \mu I_n$ has at least $q-1$ negative eigenvalues.

It will then follow that $\lambda_{j-1} - \mu_j \geq 0$ and $\mu_j - \lambda_{j+1} \geq 0$ for any $j = 1, \dots, n$. To prove (a) and (b), let $\widetilde{Z} = D_2 Z D_2 - \mu I_{n-2}$,

$$S = \begin{pmatrix} I_2 & -D_1 Y D_2 \widetilde{Z}^{-1} \\ 0 & I_{n-2} \end{pmatrix} \quad \text{and} \quad \widetilde{S} = \begin{pmatrix} I_2 & -\widetilde{D}_1 Y D_2 \widetilde{Z}^{-1} \\ 0 & I_{n-2} \end{pmatrix}.$$

Furthermore, set

$$C = Y D_2 \widetilde{Z}^{-1} D_2 Y^t, \quad B = D_1 X D_1 - \mu I_2 - D_1 C D_1 \quad \text{and} \quad \widetilde{B} = I_2 - \mu I_2 - \widetilde{D}_1 C \widetilde{D}_1.$$

Then

$$S(\mathcal{L}(G) - \mu I_n) S^t = B \oplus \widetilde{Z} \quad \text{and} \quad \widetilde{S}(\mathcal{L}(H) - \mu I_n) \widetilde{S}^t = \widetilde{B} \oplus \widetilde{Z}.$$

Evidently, condition (a) fails if and only if $\widetilde{B}$ has two positive eigenvalues but B has none; condition (b) fails if and only if $\widetilde{B}$ has two negative eigenvalues but B has none. To show that these undesirable conditions cannot happen, observe that

$$\widetilde{D}_1^{-1} \widetilde{B} \widetilde{D}_1^{-1} = (1-\mu) \operatorname{diag}(d_1 - 1, d_2 - 1) - C$$

and

$$D_1^{-1} B D_1^{-1} = \widetilde{D}_1^{-1} \widetilde{B} \widetilde{D}_1^{-1} + \begin{pmatrix} 1-\mu & -1 \\ -1 & 1-\mu \end{pmatrix} \qquad (\dagger).$$

By fact (1), we see that $\mu \in (\mu_2, \mu_1) \subseteq (0, 2)$, and thus $\begin{pmatrix} 1-\mu & -1 \\ -1 & 1-\mu \end{pmatrix}$ has eigenvalues $\eta_1 > 0 > \eta_2$, say with unit eigenvectors v_1 and v_2, respectively.

Now, if $\widetilde{B}$ has two positive eigenvalues, then so has $\widetilde{D}_1^{-1} \widetilde{B} \widetilde{D}_1^{-1}$. Using ($\dagger$) and fact (2) on $\widetilde{D}_1^{-1} \widetilde{B} \widetilde{D}_1^{-1}$, we have

$$v_1^t D_1^{-1} B D_1^{-1} v_1 = v_1^t \widetilde{D}_1^{-1} \widetilde{B} \widetilde{D}_1^{-1} v_1 + \eta_1 > 0.$$

By (2) again, we see that $D_1^{-1} B D_1^{-1}$ has at least one positive eigenvalue, and so has B.

If $\widetilde{B}$ has two negative eigenvalues, then so has $\widetilde{D}_1^{-1} \widetilde{B} \widetilde{D}_1^{-1}$. Using ($\dagger$) and fact (2) on $\widetilde{D}_1^{-1} \widetilde{B} \widetilde{D}_1^{-1}$, we have

$$v_2^t D_1^{-1} B D_1^{-1} v_2 = v_2^t \widetilde{D}_1^{-1} \widetilde{B} \widetilde{D}_1^{-1} v_2 + \eta_2 < 0.$$

By (2), $D_1^{-1} B D_1^{-1}$ has at least one negative eigenvalue, and so has B. ■

References

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