Some Matrix Techniques in Game Theory

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Abstract. In this paper, we discuss some matrix techniques in game theory. In particular, we give a short proof of the fact that equilibrium pairs of a two-person general-sum game can be found by solving certain systems of linear inequalities, and hence some standard linear programming packages such as LINDO or Maple can be used to do the computation. The technique is also used to study evolutionary games and auction games. In the former case, additional techniques are used to determine evolutionary stable strategies, and it is also shown that the computation can be done by LINDO.

Keywords: payoff matrices, two-person general-sum game, bimatrix game, evolutionary game, evolutionary stable strategy.

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1. Introduction Many real-life situations in economics, business, politics, and even evolutionary biology can be modeled as games, in which players with conflicting interests interact and try to maximize their *payoffs*, say, in terms of money, prestige, or satisfaction. We refer the readers to [I], [O], [T] or [W] for the general background of game theory.

In this paper, we focus on (finite) *two-player general-sum games*, in which the two players' interests need not be directly opposed; that is, the payoffs to both players do not necessarily sum to zero. For example, in the auction game that we study in section 4, each of the two players has a choice of finitely many *pure strategies*, which correspond to the different dollar amounts that they may bid. Suppose a player can choose to bid 2 dollars or 3 dollars. He or she can also choose the *mixed strategy* of bidding 2 dollars with probability p, and 3 dollars with probability $(1-p)$, where $0 \leq p \leq 1$. Both players would try to get the object with the lowest price relative to their evaluations of it.

Mathematically, if the two players, I and II, have m and n pure strategies respectively, I's mixed strategies are given by a nonnegative column vector $x = (x_1, \dots, x_m)^t$ where x_i is the probability of playing the i-th pure strategy; II's mixed strategies are given by a nonnegative column vector $y = (y_1, \dots, y_n)^t$ where y_j is the probability of playing the j-th pure strategy. Obviously, $\sum_{i=1}^m x_i = \sum_{j=1}^n y_j = 1$. Of course, a vector x (respectively, y) with only one non-zero entry, which equals to 1, corresponds to a pure strategy of player I (respectively, player II).

Suppose the payoffs of player I and II are given by the pair (a_{ij}, b_{ij}) when they use pure strategies i and j, respectively, for $1 \leq i \leq m$ and $1 \leq j \leq n$. Then the two $m \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ are called the *payoff matrices* of the game. If x and y are the mixed strategies chosen by I and II, then their payoffs will be computed by $\mathcal{A}(x,y) = x^t A y$ and $\mathcal{B}(x,y) = x^t B y$, respectively. Because of this representation, a two-player general-sum game is also called a *bimatrix game*.

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A pair of mixed strategies (x^*, y^*) is an *equilibrium pair* if

$$\mathcal{A}(x, y^*) \le \mathcal{A}(x^*, y^*) \quad \text{and} \quad \mathcal{B}(x^*, y) \le \mathcal{B}(x^*, y^*),$$

for any choices of mixed strategies x and y. Intuitively, an equilibrium pair is a set of strategies from which neither player will deviate unilaterally; that is, given the strategy chosen by player I, player II does not wish to change his strategy, and vice versa. It is one of the broadest non-cooperative solution concepts for a bimatrix game.

Although every bimatrix game is guaranteed to have at least one equilibrium pair by a theorem of Nash (e.g., see [T, p.57]), computing all the equilibrium pairs for a large game is rather difficult (e.g., [Wi]), and it is usually not discussed in standard references if $m, n \ge 3$ (e.g., [I], [O], [T] and [W]). In the next section, we use elementary linear algebra technique to show that such a problem can be reduced to solving a certain system of linear inequalities. As a result, one may use standard linear programming packages such as LINDO or Maple to do the computation. In sections three and four, we apply the result to analyse equilibrium strategies of evolutionary games and auction games. In the former case, additional techniques are developed to determine evolutionary stable strategies (see section 3 for the precise definitions). Furthermore, we analyse the problems for implementing the computation procedures, and discuss how one can introduce the techniques in a game theory course at the undergraduate level to enhance the understanding of the subject.

For $k = m$ or n, we shall use $\mathbf{P}_k$ to denote the set of vectors in $\mathbb{R}^k$ corresponding to all possible probability vectors of mixed strategies, i.e., vectors with nonnegative entries that sum to 1. For $k = m$ or n, we let $\{e_1^{(k)}, \ldots, e_k^{(k)}\}$ be the standard basis of $\mathbb{R}^k$, and let $e^{(k)} \in \mathbb{R}^k$ be the vector with all entries equal to one. We shall simply write e_i instead of $e_i^{(k)}$ if the size of e_i is clear from the context. Define the set of equilibrium pairs as

$$E(A, B) = \left\{ (x, y) \in \mathbf{P}_m \times \mathbf{P}_n : \begin{array}{l} x^t A y \ge \tilde{x}^t A y \ \text{ for any } \ \tilde{x} \in \mathbf{P}_m \\ \\ x^t B y \ge x^t B \tilde{y} \ \text{ for any } \ \tilde{y} \in \mathbf{P}_n \end{array} \right\}.$$

If a matrix X has k columns and $1 \le i_1 < \ldots < i_t \le k$, we denote by $X(i_1, \cdots, i_t)$ the submatrix of X formed by its $i_1, \ldots, i_t$ columns.

2. Computation of Equilibrium Pairs Let (A, B) be a pair of $m \times n$ payoff matrices for player I and player II in a bimatrix game. In the following, we show that the computation of equilibrium pairs can be reduced to a problem of solving systems of linear inequalities.

Theorem 2.1. *Let $x = x_{i_1} e_{i_1} + \cdots + x_{i_p} e_{i_p} \in \mathbf{P}_m$, with $x_{i_k} > 0$ for all k, and $y = y_{j_1} e_{j_1} + \cdots + y_{j_q} e_{j_q} \in \mathbf{P}_n$, with $y_{j_l} > 0$ for all l. Then $(x, y) \in E(A, B)$ if and only if*

$$\left[\frac{1}{p} e^{(m)} (e_{i_1} + \cdots + e_{i_p})^t - I_m \right] A(j_1, \cdots, j_q) \begin{pmatrix} y_{j_1} \\ \vdots \\ y_{j_q} \end{pmatrix} \ge \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{1a}$$

and

$$\left[\frac{1}{q}e^{(n)}(e_{j_1}+\cdots+e_{j_q})^t - I_n\right] B^t(i_1,\cdots,i_p)\begin{pmatrix} x_{i_1} \\ \vdots \\ x_{i_p}\end{pmatrix} \geq \begin{pmatrix} 0 \\ \vdots \\ 0\end{pmatrix}. \qquad (1b)$$

Proof. Let $x = x_{i_1}e_{i_1}+\cdots+x_{i_p}e_{i_p} \in \mathbf{P}_m$, with $x_{i_k} > 0$ for all k, and $y = y_{j_1}e_{j_1}+\cdots+y_{j_q}e_{j_q} \in \mathbf{P}_n$, with $y_{j_l} > 0$ for all l. Suppose $Ay = (c_1,\ldots,c_m)^t$ and $x^tB = (d_1,\ldots,d_n)$. Then $(x,y) \in E(A,B)$ if and only if

$$c_{i_1} = \cdots = c_{i_p} \geq c_i \ \text{ for all } \ i = 1,\ldots,m, \qquad (2a)$$

and

$$d_{j_1} = \cdots = d_{j_q} \geq d_j \ \text{ for all } \ j = 1,\ldots,n. \qquad (2b)$$

Note that $(2a)$ and $(2b)$ hold if and only if

$$\frac{1}{p}(c_{i_1}+\cdots+c_{i_p})e^{(m)} - \begin{pmatrix} c_1 \\ \vdots \\ c_m\end{pmatrix} \geq \begin{pmatrix} 0 \\ \vdots \\ 0\end{pmatrix}, \qquad (3a)$$

and

$$\frac{1}{q}(d_{j_1}+\cdots+d_{j_q})e^{(n)} - \begin{pmatrix} d_1 \\ \vdots \\ d_n\end{pmatrix} \geq \begin{pmatrix} 0 \\ \vdots \\ 0\end{pmatrix}. \qquad (3b)$$

One easily checks that the left hand side of $(3a)$ is the same as the left hand side of $(1a)$, and that the left hand side of $(3b)$ is the same as the left hand side of $(1b)$. The result follows. ■

In application of Theorem 2.1, one may permute the rows and columns of A and B, i.e., replacing (A,B) by (RAS, RBS) for some suitable permutation matrices R and S, so that $(i_1,\ldots,i_p) = (1,\ldots,p)$ and $(j_1,\ldots,j_q) = (1,\ldots,q)$.

Theorem 2.1 can be used for any of the following:

(a) Determine whether a given $(x,y) \in \mathbf{P}_m \times \mathbf{P}_n$ is indeed an equilibrium pair by checking whether it falls within the feasible set of the system (1).

(b) Determine whether there exists a pair $(x,y) \in E(A,B)$ with prescribed supports, i.e., positions of positive entries in the vectors x and y, respectively.

(c) In principle, one can find all $(x,y) \in E(A,B)$ by the following algorithm.

For any non-empty $P = \{i_1,\ldots,i_p\} \subseteq \{1,\ldots,m\}$ and $Q = \{j_1,\ldots,j_q\} \subseteq \{1,\ldots,n\}$, determine the feasible set of the system $(1a)$ and $(1b)$ with the additional constraints:

$$x_{i_1},\ldots,x_{i_p} > 0, \ x_{i_1}+\cdots+x_{i_p} = 1, \quad y_{j_1},\ldots,y_{i_q} > 0, \ y_{j_1}+\cdots+y_{j_q} = 1.$$

So, one has to solve $(2^m-1)(2^n-1)$ so many systems. Clearly, the computation becomes very involved when each player has a large number of pure strategies. One may improve the efficiency of the algorithm slightly as follows.

For each nonempty subset $P = \{i_1, \ldots, i_p\} \subseteq \{1, \ldots, m\}$, determine those $y \in \mathbf{P}_n$ that satisfy (1a) with $(j_1, \ldots, j_q) = (1, \ldots, n)$. One can then focus on those y and the subsets $Q = \{j_1, \ldots, j_q\}$ of $\{1, \ldots, n\}$ containing indices corresponding to the positive entries of y to solve (1b). As a result, for each non-empty subset $P \subseteq \{1, \ldots, m\}$, one may not need to consider all the subsets of $\{1, \ldots, n\}$ in the computation.

Even the modified algorithm mentioned above will improve the efficiency of the computational procedures, it is still rather difficult to determine all equilibrium pairs. Linear programming packages such as LINDO and Maple are unable to find the feasible set of the system; they can only determine whether a given set of constraints is feasible.

On the other hand, in classroom teaching, one may use the modified algorithm to study equilibrium pairs of bimatrix games with $m, n \geq 3$ to enhance the understanding of the subject. For example, assume $(m, n) = (3, 4)$. One can first identify those equilibrium pairs for which P is a singleton. For $P = \{1, 2, 3\}$, one only need to focus on those $y \in \mathbf{P}_4$ for which $Ay = (c, c, c)^t$ for some $c \in \mathbb{R}$. So, one is left with the cases when $P = \{i_1, i_2\}$ has two elements. In each of these cases, one can solve (1a) to identify those y so that $Ay = (c_1, c_2, c_3)^t$ so that $c_{i_1} = c_{i_2}$ are the largest entries in the vector, and then proceed to solve the system (1b).

3. Evolutionary Games An interesting application of game theory is to model animal behaviour evolves from generation to generation (e.g., see [O] and [W]). In such a model, one may assume that certain animal can have n types of behaviour, say, type 1 to type n. When a type i animal encounters a type j animal, it will get a reward (payoff) of a_{ij} units (in terms of food, territory, etc.). One would consider such a model as a bimatrix game with payoff matrices A and A^t, and call it an *evolutionary game*. A mixed strategy $x \in \mathbf{P}_n$ can be viewed as the proportion of these various types of animals in the population, i.e., $x_i \geq 0$ is the fraction of type i animals in the system. One may also regard $x \in \mathbf{P}_n$ as a *genotype* of the animal, i.e., there is a probability of x_i for a newly born animal to have type i behaviour. Then the expected payoff of $x \in \mathbf{P}_n$ is computed by x^tAx. We say that x is an *evolutionary stable strategy* (abbreviate to ESS) if for any $y \in \mathbf{P}_n$ there exists $\delta > 0$ such that

$$x^tA(ry + (1-r)x) > y^tA(ry + (1-r)x)$$

for any $0 < r < \delta$. Roughly speaking, this condition ensures that if a small proportion (no more than δ) of a different mutant genotype $y \in \mathbf{P}_n$ enters the system, the expected payoff of the existing genotype x will be higher than that of the mutant genotype y. Thus the system will have a tendency to return to the original sate.

It is known (e.g., see [O] and [W]) that a genotype $x \in \mathbf{P}_n$ is ESS if and only if the following two conditions hold:

(i) $x^tAx \geq y^tAx$ for any $y \in \mathbf{P}_n$,

(ii) if $y \in \mathbf{P}_n$ is such that $y \neq x$ and $x^tAx = y^tAx$, then $x^tAy > y^tAy$.

One easily sees that an $x \in \mathbf{P}_n$ satisfying condition (i) if and only if (x, x) is an equilibrium pair of the bimatrix game (A, A^t). Such an x is called an *equilibrium genotype* of the evolutionary game. We shall use Theorem 2.1 and some additional matrix techniques to derive some effective schemes for checking conditions (i) and (ii).

In the following, We shall use $\{e_1, \ldots, e_n\}$ to denote the standard basis of $\mathbb{R}^n$, and let $e \in \mathbb{R}^n$ be the vector with all entries equal to one. If a matrix X has k columns and $1 \le i_1 < \ldots < i_t \le k$, we shall continue to denote by $X(i_1, \cdots, i_t)$ the submatrix of X formed by its $i_1, \ldots, i_t$ columns.

By Theorem 2.1, we have the following result.

Theorem 3.1. *A mixed strategy $x = x_{i_1} e_{i_1} + \cdots + x_{i_p} e_{i_p} \in \mathbf{P}_n$ with $x_{i_k} > 0$ for all $1 \le k \le p$ is an equilibrium genotype of the evolutionary game (A, A^t) if and only if*

$$\left[\frac{1}{p} e(e_{i_1} + \cdots + e_{i_p})^t - I_n\right] A(i_1, \cdots, i_p) \begin{pmatrix} x_{i_1} \\ \vdots \\ x_{i_p} \end{pmatrix} \ge \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{4}$$

Notice that it is relatively easy to determine all the equilibrium genotypes in an evolutionary game. One only need to solve the system (4) for each non-empty subset $P = \{i_1, \ldots, i_p\}$ of $\{1, \ldots, n\}$ with the additional constraints that $x_{i_1}, \ldots, x_{i_p} > 0$ and $x_{i_1} + \cdots + x_{i_p} = 1$. Moreover, the study of condition (i) is useful in reducing the number of cases needed to be checked for condition (ii) as shown in the following.

Theorem 3.2. *Suppose $\tilde{x} = \tilde{x}_{i_1} e_{i_1} + \cdots + \tilde{x}_{i_p} e_{i_p} \in \mathbf{P}_n$ with $\tilde{x}_{i_k} > 0$ for all $1 \le k \le p$ is an equilibrium genotype of the evolutionary game (A, A^t). If $x \in \mathbf{P}_n$ is such that $x \ne \tilde{x}$ and the $i_1, \ldots, i_p$ entries of x are all positive, then x cannot satisfy condition (ii) and hence is not an ESS.*

Proof. Suppose x and $\tilde{x}$ satisfy the hypotheses of the theorem. If $Ax = (c_1, \ldots, c_n)^t$ is an equilibrium genotype, then $c_{i_1} = \cdots = c_{i_p} \ge c_j$ for any $1 \le j \le n$. Thus $\tilde{x}^t A x = x^t A x$. However, since $\tilde{x}$ is also an equilibrium genotype, it is impossible to have $x^t A \tilde{x} > \tilde{x}^t A \tilde{x}$, and thus condition (ii) does not hold. ∎

Now, suppose $x = \sum_{k=1}^p x_{i_k} e_k$ is an equilibrium genotype with $x_{i_k} > 0$ for all k such that no other equilibrium genotype $\tilde{x}$ has supports lying in $\{i_1, \ldots, i_p\}$. For simplicity, we let $(i_1, \ldots, i_p) = (1, \ldots, p)$. Otherwise, we may permute the rows and the corresponding columns of A to achieve that. [Note that for a general bimatrix game, we may replace (A, B) by (RAS, RBS) for suitable permutation matrices R and S. For an evolutionary game (A, A^t) one should use (RAR^t, RA^tR^t) for a suitable permutation matrix R to preserve the structure.] To check whether condition (ii) holds for such an x, let $Ax = (c_1, \ldots, c_n)^t$. Since x is a equilibrium point, $c_1 = \cdots = c_p \ge c_i$ for all i. It is possible that there are $j > p$ such that $c_j = c_p$. Again, we may apply a permutation to rows and the corresponding columns of A and assume that $c_1 = \cdots = c_q > c_j$ for all $j > q$. Standard method (e.g., see [T, Chapter 8]) of testing ESS requires checking $(x - y)^t A(x - y) > 0$

for all $x = (x_1, \ldots, x_p, 0, \ldots, 0)^t, y = (y_1, \ldots, y_q, 0, \ldots, 0)^t \in \mathbf{P}_n$. If $q \le p+1$, this can be done effectively (e.g., see [T, Problems 8.3 - 8.4]). However, if $q > p+1$, there is no easy theoretical technique or computer package that can test this condition. To get around this problem, we have the following result.

Theorem 3.3. *Suppose $x = (x_1, \ldots, x_p, 0, \ldots, 0)^t$ is an equilibrium genotype of the evolutionary game (A, A^t) such that $Ax = (c_1, \ldots, c_n)^t$ satisfies $c_1 = \cdots = c_q > c_j$ for $j > q$. Let $C = B + B^t$, where B is the leading principal $q \times q$ submatrix of $(ex^t - I_n)A$. Then x is an ESS if and only if $y^tCy > 0$ for any $y \in \mathbb{R}^q$ with nonnegative entries that sum up to 1.*

Proof. By the given condition, $y \in \mathbf{P}_n$ satisfies $x^tAx = y^tAx$ if and only if y is of the form $(y_1, \ldots, y_q, 0, \ldots, 0)^t$. For such a y, we need to check whether

$$0 < x^tAy - y^tAy = y^tex^tAy - y^tAy = y^t[(ex^t - I_n)A]y.$$

Since y only has support at the first q positions, and $z^tXz = z^t(X + X^t)z/2$ for any $X \in \mathbb{R}^{n\times n}$ and $z \in \mathbb{R}^n$. We get the desired conclusion. ■

Our condition in Theorem 3.3 is simple to check. For example, the linear programming package LINDO can solve the optimization problem

$$\min y^tCy \quad \text{subject to } y \in \mathbf{P}_n.$$

Note also that a real symmetric matrix C satisfies $y^tCy > 0$ for all nonzero nonnegative vector y is known as *copositive* matrix, and is quite well-studied in matrix theory literature (e.g., see [A] and its references).

4. Auction Games Auction games, in which one or more objects are sold to the highest bidder(s), present another opportunity to apply Theorem 2.1, as well as to investigate the special structure of the auction payoff matrices. In this section, we assume that only one object is sold under Dutch auction rules. That is, the auctioneer starts the bidding at a high price (far higher than the expected selling price) and lowers the price until someone agrees to buy the object at the current price. In effect, therefore, the highest bidder gets the object at a price equal to the highest bid. In our model, there are two bidders who value the object at $v_1 > 0$ and $v_2 > 0$ respectively. We further specify that v_1 and v_2 are integers, and that $v_1 \ge v_2$. We assume that if both players bid the same amount, then they each have a $\frac{1}{2}$ chance of getting the object. Finally, we only allow integer bids. This last restriction is reasonable because any real-world auction must set a minimum allowable increment for bids. Even in the most extreme case, the minimum increment is one cent.

Each player has $(b+1)$ pure strategies, where the k-th pure strategy is to bid $(k-1)$ dollars. Note that $k = 1, \ldots, b+1$, where b is the price at which the auctioneer starts the bidding. It has been pointed out, however, that we only need concern ourselves with bids less than or equal to v_1, as neither player will bid more than this amount. See [T, p.102] for a proof. Thus, the payoff matrices A and B are $(v_1+1) \times (v_1+1)$, where the (i, j) entry is the payoff from I bidding $(i-1)$ and II bidding $(j-1)$.

In the following, we shall use $\{e_0, \ldots, e_{v_1}\}$ to denote the standard basis for $\mathbb{R}^{v_1+1}$, and always assume $x = (x_0, \ldots, x_{v_1})^t, y = (y_0, \ldots, y_{v_1})^t \in \mathbf{P}_{v_1+1}$ to be the mixed strategies of the auctioneers. We shall use $\mathcal{A}(x, y)$ and $\mathcal{B}(x, y)$ to denote the payoff of the auctioneers using the mixed strategies x and y, respectively.

Theorem 4.1. *If $v_1 \geq v_2 + 2$, then equilibrium pairs (x, y) of the form $x = x_{i_1}e_{i_1} + \cdots + x_{i_p}e_{i_p} \in \mathbf{P}_{v_1+1}$, with $x_{i_k} > 0$ for all k, and $y = y_{j_1}e_{j_1} + \cdots + y_{j_q}e_{j_q} \in \mathbf{P}_{v_1+1}$, with $y_{j_l} > 0$ for all l, satisfy one of the following:*

$$0 < i_1 < \cdots < i_p \leq j_q + 1 \leq v_2 \tag{5a}$$

$$x = x_{v_2}e_{v_2} \;\text{ or }\; x_{v_2}e_{v_2} + x_{v_2+1}e_{v_2+1}, \;\text{ and }\; j_q \leq v_2 \tag{5b}$$

$$x = x_{i_1}e_{i_1} \;\text{ with }\; i_1 > v_2, \;\text{ and }\; j_q \leq i_1 - 1. \tag{5c}$$

Proof. The payoff matrices A and B are

$$A = \begin{pmatrix} \frac{v_1}{2} & 0 & \cdots & 0 & 0 \\ (v_1 - 1) & \frac{(v_1-1)}{2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & \frac{1}{2} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \frac{v_2}{2} & (v_2 - 1) & \cdots & (v_2 - v_1 + 1) & (v_2 - v_1) \\ 0 & \frac{(v_2-1)}{2} & \cdots & (v_2 - v_1 + 1) & (v_2 - v_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{(v_2-v_1+1)}{2} & (v_2 - v_1) \\ 0 & 0 & \cdots & 0 & \frac{(v_2-v_1)}{2} \end{pmatrix}.$$

We can immediately eliminate from consideration the last two strategies (bid v_1 and $v_1 - 1$) for player II, as well as the last strategy (bid v_1) for player I, as they are strictly dominated. Specifically, $\mathcal{B}(x, y_0e_0) > \mathcal{B}(x, y_{v_1}e_{v_1})$ for all mixed strategies x. Once y_0e_0 is removed, $\mathcal{A}(x_{v_1-1}e_{v_1-1}, y) > \mathcal{A}(x_{v_1}e_{v_1}, y)$ for all y. Once $x_{v_1}e_{v_1}$ is also removed, $\mathcal{B}(x, y_0e_0) > \mathcal{B}(x, y_{v_1-1}e_{v_1-1})$ for all x, and we can remove $y_{v_1-1}e_{v_1-1}$ as well. When strictly dominated strategies are thus removed, the equilibrium pairs of the reduced game are identical to the equilibrium pairs of the original game. See for example, [T, p.83] for a proof. In other words, $i_p \leq v_1 - 1$ and $j_q \leq v_1 - 2$.

Now, we observe that

$$Ay = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{j_q} \\ c_{j_q+1} \\ c_{j_q+2} \\ \vdots \\ c_{v_1} \end{pmatrix} = \begin{pmatrix} v_1 \\ v_1 - 1 \\ \vdots \\ v_1 - j_q \\ v_1 - j_q - 1 \\ v_1 - j_q - 2 \\ \vdots \\ 0 \end{pmatrix} \circ \begin{pmatrix} \frac{y_0}{2} \\ y_0 + \frac{y_1}{2} \\ \vdots \\ y_0 + \cdots + \frac{y_{j_q}}{2} \\ y_0 + \cdots + y_{j_q} \\ y_0 + \cdots + y_{j_q} \\ \vdots \\ y_0 + \cdots + y_{j_q} \end{pmatrix},$$

where $\circ$ denotes the Schur (entrywise) product. Note that c_0 is never the largest entry in Ay: If $y_0 = 0$, then there is some other $y_k > 0$ for $0 < k \leq j_q$; in this case, $c_k > c_0$. If $y_0 > 0$, then $c_1 > c_0$ as long as $v_1 > 2$, which always holds if $v_1 \geq v_2 + 2$. Note also that $c_{j_q+1} > c_{j_q+k}$ for all $k > 1$.

Thus, equilibrium pairs of the form $x = x_{i_1}e_{i_1} + \cdots + x_{i_p}e_{i_p} \in \mathbf{P}_{v_1+1}$, with $x_{i_k} > 0$ for all k, and $y = y_{j_1}e_{j_1} + \cdots + y_{j_q}e_{j_q} \in \mathbf{P}_{v_1+1}$, with $y_{j_l} > 0$ for all l, must satisfy

$$0 < i_1 < \cdots < i_p \leq j_q + 1 \quad \text{and} \quad 0 \leq j_1 < \cdots < j_q \leq v_1 - 2. \tag{6}$$

Player II's situation presents the following analysis:

(a) If $i_1 < v_2$, then $x^tB = (d_0, \ldots, d_{v_1})$ satisfies $d_{v_2-1} > d_{v_2} = 0$. So we have $j_q < v_2$. Substituting into the first part of (6) yields $0 < i_1 < \cdots < i_p \leq j_q + 1 \leq v_2$. Obviously, this result corresponds to ($5a$) above.

(b) If $i_1 = v_2$, then $x^tB = (d_0, \ldots, d_{v_1})$ satisfies $d_0 = \cdots = d_{v_2} = 0 > d_{v_2+1}$, which implies that $j_q \leq v_2$. Substituting into the first part of (6) gives $v_2 = i_1 < i_p \leq j_q + 1 \leq v_2 + 1$. Hence, equilibrium pairs satisfy $x = x_{v_2}e_{v_2}$ or $x = x_{v_2}e_{v_2} + x_{v_2+1}e_{v_2+1}$, and $j_q \leq v_2$. This result corresponds to ($5b$) above.

(c) If $i_1 > v_2$, then $x^tB = (d_0, \ldots, d_{v_1})$ satisfies $d_0 = \cdots = d_{i_1-1} = 0 > d_{i_1}$. Therefore, $j_q \leq i_1 - 1$. From the first part of (6), we have $i_p \leq j_q + 1 \leq i_1$, which implies $p = 1$ and $i_1 = j_q + 1$. Thus, equilibrium pairs satisfy $x = x_{i_1}e_{i_1}$ and $j_q \leq i_1 - 1$, which corresponds to ($5c$) above. ■

Example 4.2. Let $v_1 = 4$ and $v_2 = 2$. Then Theorem 4.1 tells us that all equilibrium pairs of the form $x = x_{i_1}e_{i_1} + \cdots + x_{i_p}e_{i_p} \in \mathbf{P}_5$, with $x_{i_k} > 0$ for all k, and $y = y_{j_1}e_{j_1} + \cdots + y_{j_q}e_{j_q} \in \mathbf{P}_5$, with $y_{j_l} > 0$ for all l, must satisfy one of the following:

(a) $x_1 + x_2 = 1$ and $y_0 + y_1 = 1$;
(b) $x = x_2e_2$ or $x = x_2e_2 + x_3e_3$ and $\sum_{j=0}^{2} y_j = 1$;
(c) $x = x_3e_3$ and $\sum_{j=0}^{2} y_j = 1$.

We can use Theorem 2.1 on the reduced sets P and Q to find the equilibrium pairs

(a) $x = x_2e_2$ and $y = y_{j_1}e_{j_1} + \cdots + y_{j_q}e_{j_q}$ such that $\sum_{j=0}^{2} y_j = 1$ and $y_2 + \frac{y_1}{2} \geq y_0$;
(b) $x = x_{i_1}e_{i_1} + \cdots + x_{i_p}e_{i_p}$ such that $x_2 + x_3 = 1$ and $y = y_2e_2$.

Theorem 4.1 can be extended to the case where $v_1 = v_2 + 1$; however, it is no longer very helpful. First, (6) must be modified to include $j_q = v_1 - 1$, as this strategy is no longer strictly dominated. Thus, (6) becomes

$$0 < i_1 < \cdots < i_p \leq j_q + 1 \quad \text{and} \quad 0 \leq j_1 < \cdots < j_q \leq v_1 - 1. \tag{6$'$}$$

Assuming that $v_1 > 2$, we can substitute into ($5a$) to obtain

$$0 < i_1 < \cdots < i_p \leq j_q + 1 \leq v_1 - 1$$

But this result is not much more helpful than the initial result in ($6'$). (If $v_1 \leq 2$, then ($6'$) must be further expanded to consider $i_1 = 0$.) For completeness, we note that if $v_1 = v_2 + 1$, case (b) ($i_1 = v_2$) reduces to $x = x_{v_2}e_{v_2}$ and $j \leq v_2$, since $x = x_{v_2+1}e_{v_2+1}$ is strictly dominated. Furthermore, case (c) ($i_1 > v_2$) never occurs for the same reason.

For the case where $v_1 = v_2$, Theorem 4.1 still holds. However, the theorem is not very useful, as there are no strictly dominated strategies to remove. Thus, the results of this section are more useful when there is a relatively large gap between v_1 and v_2. If the gap is relatively small, then case (a) ($i_1 < v_2$) still contains most of the (x, y) that were under consideration before the application of Theorem 4.1.

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