DAVIS–WIELANDT SHELLS OF NORMAL OPERATORS

CHI-KWONG LI AND YIU-TUNG POON

Dedicated to Professor Hans Schneider for his 80th birthday.

Abstract. For a finite-dimensional operator A with spectrum $\sigma(A)$, the following conditions on the Davis–Wielandt shell $DW(A)$ of A are equivalent:

(a) A is normal.

(b) $DW(A)$ is the convex hull of the set $\{(\lambda, |\lambda|^2) : \lambda \in \sigma(A)\}$.

(c) $DW(A)$ is a polyhedron.

These conditions are no longer equivalent for an infinite-dimensional operator A. In this note, a thorough analysis is given for the implication relations among these conditions. From the main result, one can deduce the equivalent conditions (a) — (c) for an finite-dimensional operator A, and show that the Davis–Wielandt shell cannot be used to detect normality for infinite-dimensional operators.

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1. Introduction

Let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators acting on the Hilbert space $\mathcal{H}$. We identify $\mathcal{B}(\mathcal{H})$ with the algebra M_n of $n \times n$ complex matrices if $\mathcal{H}$ has dimension n. The *numerical range* of $A \in \mathcal{B}(\mathcal{H})$ is defined by

$$W(A) = \{\langle Ax, x\rangle : x \in \mathcal{H}, \langle x, x\rangle = 1\};$$

see [7, 9, 10]. The numerical range is useful in studying matrices and operators. In particular, the geometrical properties of $W(A)$ often provide useful information on the algebraic and analytic properties of A. For instance, $W(A) = \{\mu\}$ if and only if $A = \mu I$; $W(A) \subseteq \mathbb{R}$ if and only if $A = A^*$; $W(A)$ has no interior point if and only if there are $a, b \in \mathbb{C}$ with $a \neq 0$ such that $aA + bI$ is self-adjoint. Moreover, there are nice connections between $W(A)$ and the spectrum $\sigma(A)$ of A. For example, the closure of $W(A)$, denoted by $\mathbf{cl}\,(W(A))$, always contains $\sigma(A)$. If A is normal, then $\mathbf{cl}\,(W(A)) = \mathbf{conv}\sigma(A)$, where $\mathbf{conv}\sigma(A)$ denotes the convex hull of $\sigma(A)$. However, the converse is not true; for example, see Problem 10 in [10, p.14].

Motivated by theoretical study and applications, researchers have considered many generalizations of the numerical range; see for example [10, Chapter 1]. One of these generalizations is the *Davis–Wielandt shell* of $A \in \mathcal{B}(\mathcal{H})$ defined by

$$DW(A) = \{(\langle Ax, x\rangle, \langle Ax, Ax\rangle) : x \in \mathcal{H}, \langle x, x\rangle = 1\};$$

see [5, 6, 11]. Evidently, the projection of the set $DW(A)$ on the first coordinate is the classical numerical range. So, $DW(A)$ captures more information about the operator A. For example, in the finite-dimensional case, normality of operators can be completely determined by the geometrical shape of their Davis–Wielandt shells. In particular, the following conditions are equivalent for $A \in M_n$; see [10, Section 1.8] and the references therein. (See also Corollary 2.4.)

(a) A is normal.

(b) $DW(A)$ is the convex hull of the set $\{(\lambda, |\lambda|^2) : \lambda \in \sigma(A)\}$.

(c) $DW(A)$ is a polyhedron.

These conditions are no longer equivalent for an infinite-dimensional operator A. We will give a thorough analysis of the implications among these conditions. In particular, it is shown that $DW(A)$ cannot be used to detect normality for infinite-dimensional operators.

2. Results and proofs

Denote by $\mathbf{cl}\,(S)$ and ∂S the closure and the boundary of a set S. Let $A \in \mathcal{B}(H)$. The *point spectrum* of A is the set $\sigma_p(A)$ of eigenvalues of A. The *approximate point spectrum* of A is the set $\sigma_a(A)$ of complex number $\lambda \in \mathbb{C}$ such that the exists a sequence of unit vectors $\{x_n\}_1^\infty$ in $\mathcal{H}$ such that $\lim_{n\to\infty} \|(\lambda I - A)x_n\| = 0$.

Theorem 2.1. *Suppose $A \in \mathcal{B}(\mathcal{H})$ is an infinite-dimensional normal operator. Then*

$$DW(A) \subseteq \mathbf{conv}\{(\lambda, |\lambda|^2) : \lambda \in \sigma(A)\} = \mathbf{cl}\,(DW(A))\,. \tag{2.1}$$

Proof. Note that $\sigma(A)$ is compact, and hence $\mathbf{conv}\{(\lambda, |\lambda|^2) : \lambda \in \sigma(A)\}$ is a compact convex set.

Since A is normal, we have $\sigma(A) = \sigma_a(A)$; see [8, Theorem 31.2]. By the Spectral Theorem [4], $A = \int z dE(z)$ for some spectral measure E on the Borel subsets of $\sigma(A)$. Given a unit vector $x \in \mathcal{H}$, we have

$$\langle Ax, x\rangle = \int z dE_{x,x}(z) \quad \text{and} \quad \langle A^*Ax, x\rangle = \int |z|^2 dE_{x,x}(z).$$

Since $E_{x,x}$ is a probability measure on $\sigma(A)$, the first inclusion in (2.1) follows.

Now, suppose $\lambda \in \sigma(A) = \sigma_a(A)$. Then there is a sequence of unit vectors $\{x_n\}_1^\infty$ such that $\lim_{n\to\infty} \|(A-\lambda I)x_n\| = 0$. Since A is normal, we have $\lim_{n\to\infty} \|(A^* - \overline{\lambda}I)x_n\| = \lim_{n\to\infty} \|(A-\lambda I)x_n\| = 0$. From

$$(A^*A - |\lambda|^2 I)x_n = A^*(A-\lambda I)x_n + \lambda(A^* - \overline{\lambda}I)x_n,$$

we have

$$\lim_{n\to\infty} \|(A^*A - |\lambda|^2 I)x_n\| = 0.$$

Hence, $\lim_{n\to\infty} \langle Ax_n, x_n\rangle = \lambda$ and $\lim_{n\to\infty} \langle A^*Ax_n, x_n\rangle = |\lambda|^2$. Consequently,

$$\{(\lambda, |\lambda|^2) : \lambda \in \sigma(A)\} \subseteq \mathbf{cl}\,(DW(A))$$

and the equality in (2.1) follows. □

The following examples shows that $DW(A)$ may not be close for an infinite-dimensional operator A.

Example 2.2. *Let* $A = \operatorname{diag}(1, 1/2, 1/3, \dots)$. *Then* $\sigma(A) = \{0\} \cup \{1/n : n \geq 1\}$, $DW(A)$ *is not closed and* $(0,0) \in \mathbf{cl}\,(DW(A)) \setminus DW(A)$.

As mentioned before, if $A \in M_n$ satisfies $\mathbf{conv}\{(\lambda, |\lambda|^2) : \lambda \in \sigma(A)\} = DW(A)$, then A is normal. It is easy to show that if $A \in \mathcal{B}(\mathcal{H})$ is normal with finite spectrum then $DW(A) = \mathbf{conv}DW(A)$ is the convex polyhedron

$$\mathbf{conv}\{(\lambda, |\lambda|^2) : \lambda \in \sigma(A)\}$$

in $\mathbb{C} \times \mathbb{R}$, identified with $\mathbb{R}^3$. We show that the converse is also true in Theorem 2.3. To prove the result, we need the following construction of Berberian [3]. Denote by Lim a fixed Banach generalized limit, defined for bounded sequences of complex numbers; thus for two bounded sequences of complex numbers $\{a_n\}$ and $\{b_n\}$:

(1) $\operatorname{Lim}(a_n + b_n) = \operatorname{Lim} a_n + \operatorname{Lim} b_n$.
(2) $\operatorname{Lim}(\gamma a_n) = \gamma \operatorname{Lim} a_n$.
(3) $\operatorname{Lim} a_n = \lim a_n$ whenever $\lim a_n$ exists.
(4) $\operatorname{Lim} a_n \geq 0$ whenever $a_n \geq 0$ for all n.

We note that the translation invariant property of Lim is not assumed here. Denote by $\mathcal{V}$ the set of all bounded sequences $\{x_n\}$ with $x_n \in \mathcal{H}$. Then $\mathcal{V}$ is a vector space relative to the definitions $\{x_n\} + \{y_n\} = \{x_n + y_n\}$ and $\gamma\{x_n\} = \{\gamma x_n\}$. Let $\mathcal{N}$ be the set of all sequences $\{x_n\}$ such that $\operatorname{Lim}\langle x_n, y_n\rangle = 0$ for all $\{y_n\} \in \mathcal{V}$. Then $\mathcal{N}$ is a linear subspace of $\mathcal{V}$. Denote by $\mathbf{x}$ the coset $\{x_n\}+\mathcal{N}$. The quotient vector space $\mathcal{V}/\mathcal{N}$ becomes an inner product space with the inner product $\langle \mathbf{x},\mathbf{y}\rangle = \text{Lim } \langle x_n, y_n\rangle$. Let $\mathcal{H}_0$ be the completion of $\mathcal{V}/\mathcal{N}$. If $x \in \mathcal{H}$, then $\{x\}$ denotes the constant sequence defined by x. Since $\langle \mathbf{x},\mathbf{y}\rangle = \langle x,y\rangle$ for $\mathbf{x} = \{x\}+\mathcal{N}$ and $\mathbf{y} = \{y\}+\mathcal{N}$, the mapping $x \mapsto \mathbf{x}$ is an isometric linear map of $\mathcal{H}$ onto a closed subspace of $\mathcal{H}_0$ and $\mathcal{H}_0$ is an extension of $\mathcal{H}$. For an operator $A \in \mathcal{B}(\mathcal{H})$, define

$$A_0(\{x_n\}+\mathcal{N}) = \{Ax_n\}+\mathcal{N}.$$

We can extend A_0 on $\mathcal{H}_0$, which will be denoted by A_0 also. The mapping $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}_0)$ given by $\Phi(A) = A_0$ is $*$-isomorphic and isometric such that $\sigma_a(A) = \sigma_a(A_0) = \sigma_p(A_0)$; see [3]. Using this construction, we can prove the following.

Theorem 2.3. *Let $A \in \mathcal{B}(\mathcal{H})$. Then the following conditions are equivalent if we identify $\mathbb{C}\times\mathbb{R}$ with $\mathbb{R}^3$:*

1. *$DW(A)$ is a (closed) polyhedron in $\mathbb{C}\times\mathbb{R}$.*
2. *$\mathbf{cl}\,(DW(A))$ is a polyhedron in $\mathbb{C}\times\mathbb{R}$.*
3. *A is normal with finite spectrum, i.e., there are complex numbers $a_1,\dots,a_m \in \mathbb{C}$ and an orthogonal decomposition of $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m$ such that*

$$A = a_1 I_{\mathcal{H}_1} \oplus \cdots \oplus a_m I_{\mathcal{H}_m}.$$

Proof. The implications (3) $\Rightarrow$ (1) and (1) $\Rightarrow$ (2) are clear.

We consider the implication (2) $\Rightarrow$ (3). Suppose (2) holds.

If dim $\mathcal{H} = 2$, then $W(A)$ is a polygon in $\mathbb{C}$. Hence, A is normal.

Suppose dim $\mathcal{H} > 2$. Let $A = H + iG$, where H and G are self-adjoint, and let $K = A^*A$. Then $DW(A)$ can be identified with the joint numerical range

$$W(H,G,K) = \{(\langle Hx,x\rangle, \langle Gx,x\rangle, \langle Kx,x\rangle) : x \in \mathcal{H},\ \langle x,x\rangle = 1\}.$$

Note that $\mathbf{cl}\,(W(H,G,K))$ is a compact convex set in $\mathbb{R}^3$ [1], which can be obtained as the intersection of the half spaces of the form

$$\{(h,g,k) : (h,g,k)(a,b,c)^t \le \sup \sigma(aH+bG+cK)\}$$

for all unit vectors $(a,b,c)^t \in \mathbb{R}^3$. We can use the above construction of Berberian [3] to embed $\mathcal{H}$ into $\mathcal{H}_0$ and extend (H,G,K) to (H_0,G_0,K_0). Since Φ is $*$-isomorphic, $K_0 = (A^*A)_0 = A_0^*A_0$. Furthermore, we have

$$\sup \sigma(aH+bG+cK) = \sup \sigma(aH_0+bG_0+cK_0)$$

for all unit vectors $(a,b,c)^t \in \mathbb{R}^3$. So, $\mathbf{cl}\,(W(H_0,G_0,K_0)) = \mathbf{cl}\,(W(H,G,K))$ is a convex polyhedron in $\mathbb{R}^3$. Now, suppose $v = (h,g,k)$ is a vertex of the polyhedron $\mathbf{cl}\,(W(H,G,K))$. Then there is a sequence of unit vectors $\{x_n\} \in \mathcal{H}$ such that

$$(\langle Hx_n, x_n\rangle, \langle Gx_n, x_n\rangle, \langle Kx_n, x_n\rangle) \to v.$$

Regarding $x = \{x_n\} + \mathcal{N}$ as an element in $\mathcal{H}_0$, we see that

$$v = (h, g, k) = (\langle H_0 x, x\rangle, \langle G_0 x, x\rangle, \langle K_0 x, x\rangle) \in W(H_0, G_0, K_0).$$

This shows that $W(H_0, G_0, K_0)$ is closed. Since v is a vertex of the polyhedron $W(H_0, G_0, K_0)$, there are three support planes of the polyhedron with linearly independent normal vectors passing through v. Thus, there are three linearly independent unit vectors $(a_j, b_j, c_j)^t \in \mathbb{R}^3$ with $1 \le j \le 3$ such that

$$a_j(H_0 - hI) + b_j(G_0 - gI) + c_j(K_0 - kI)$$

is positive semidefinite with x as a null vector. In other words, we have

$$[a_j(H_0 - hI) + b_j(G_0 - gI) + c_j(K_0 - kI)]x = 0, \qquad j = 1, 2, 3.$$

Since $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is nonsingular, we see that

$$(H_0 - hI)x = 0, \quad (G_0 - gI)x = 0, \quad \text{and} \quad (K_0 - kI)x = 0. \tag{2.2}$$

Therefore, $A_0 x = (h + ig)x$ and $A_0^* x = (h - ig)x$. Hence, x is a reducing eigenvector for A_0. Also, we have $k = \langle K_0 x, x\rangle = \langle A_0 x, A_0 x\rangle = h^2 + g^2$. Therefore, $v = (h, g, h^2 + g^2)$ and

$$DW(A_0) \subseteq \mathbf{conv}\{(\lambda, |\lambda|^2) : \lambda \in \sigma_p(A_0)\}.$$

Hence, $DW(A_0) = \mathbf{conv}\{(\lambda, |\lambda|^2) : \lambda \in \sigma_p(A_0)\}$. Let $v_j = (h_j, g_j, h_j^2 + g_j^2)$, $j = 1, \ldots, m$, be the vertices of the polyhedron $DW(A_0)$. Then the above argument shows that for each j, $a_j = h_j + ig_j$ is a reducing eigenvalue of A_0. Let S_j be the eigenspace of A_0 associated with a_j. Then

$$A_0 = a_1 I_{S_1} \oplus \cdots \oplus a_m I_{S_m} \oplus B_0$$

such that $\{a_1, \ldots, a_m\} \cap \sigma_p(B_0) = \emptyset$ and

$$DW(B_0) \subseteq DW(A_0) = \mathbf{conv}\{(a_j, |a_j|^2) : 1 \le j \le m\}.$$

If B_0 is present, then $\sigma_a(B_0)$ is nonempty as it contains the boundary of the nonempty compact set $\sigma(B_0)$; see for example [9, Chapter 9]. But then we can find $b \in \sigma_p(B_0) = \sigma_a(B_0)$ and $b \notin \{a_1, \ldots, a_m\}$. Note that the set

$\{(\lambda, |\lambda|^2) : \lambda \in \mathbb{C}\}$ is the graph of the strictly convex function $f : \mathbb{C} \to \mathbb{R}$ defined by $f(\lambda) = |\lambda|^2$. Hence, $(b, |b|^2) \in DW(B_0)$ and

$$(b, |b|^2) \notin \mathbf{conv}\{(a_j, |a_j|^2) : 1 \le j \le m\} = DW(A_0),$$

which contradicts the fact that $DW(B_0) \subseteq DW(A_0)$. Thus, B_0 is absent, and $A_0 = a_1 I_{S_1} \oplus \cdots \oplus a_m I_{S_m}$ is normal with finite spectrum. Following the construction of A_0 from A, we see that A is normal with finite spectrum $\{a_1, \dots, a_m\}$. Thus, A has the form $a_1 I_{\mathcal{H}_1} \oplus \cdots \oplus a_m I_{\mathcal{H}_m}$ as asserted. □

If A is a finite-dimensional operator, then $\sigma(A)$ is finite and $DW(A)$ is always closed. Thus, we have the following corollary.

Corollary 2.4. *For a finite-dimensional operator A with spectrum $\sigma(A)$, the following conditions are equivalent.*

(a) *A is normal.*

(b) *$DW(A)$ is the convex hull of the set $\{(\lambda, |\lambda|^2) : \lambda \in \sigma(A)\}$.*

(c) *$DW(A)$ is a polyhedron.*

Next, we show that the Davis–Wielandt shell cannot be used to detect the normality of (infinite-dimensional) operators. In particular, there are normal and nonnormal operators having the same Davis–Wielandt shell. Constructing an example for $\mathcal{H}$ with an uncountable dimension is relatively easy. Here we present an example for an operator acting on a separable Hilbert space.

Example 2.5. *Let $\mathcal{H}$ be a Hilbert space with an orthonormal basis $\{e_n : n \ge 1\}$, and let $A \in \mathcal{B}(\mathcal{H})$ be such that $Ae_j = d_j e_j$ for $j = 1, 2, \dots,$ where $\{d_n : n \ge 1\}$ is a (countable) dense subset of $\{\lambda \in \mathbb{C} : |\lambda| \le 1\}$ containing $0, 1, -1$. Suppose $B = A \oplus C$ with $C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then A is normal and B is not normal such that*

$$\sigma(A) = \sigma(B) = \{\lambda \in \mathbb{C} : |\lambda| \le 1\}, \qquad DW(A) = DW(B),$$

and

$$\mathbf{cl}\,(DW(A)) = \mathbf{cl}\,(DW(B)) = \mathbf{conv}\{(\lambda, |\lambda|^2) : \lambda \in \sigma(B)\}.$$

We verify the assertions in the above example in the following.

Evidently, A is normal and B is not normal. It is easy to see that $\sigma(A) = \sigma(B) = \{\lambda \in \mathbb{C} : |\lambda| \le 1\}$ and

$$\mathbf{cl}\,(DW(A)) = \mathbf{conv}\{(\lambda, |\lambda|^2) : |\lambda| \le 1\}.$$

$$DW(A) = \{(z, r) : 0 \ge r < |z|^2 < 1\} \cup \mathbf{conv}\{(d_n, |d_n|^2) : n \ge 1\}\,.$$

By definition, $DW(C)$ consists of all $(z,r) \in \mathbb{C} \times \mathbb{R}$ of the form $(z,r) = (\overline{x}_1 x_2, |x_2|^2)$ for some $x_1, x_2 \in \mathbb{C}$ such that $|x_1|^2 + |x_2|^2 = 1$. Thus, if $(z,r) \in DW(C)$, we have

$$|z|^2 = r(1-r) \Leftrightarrow |z|^2 + \left(r - \frac{1}{2}\right)^2 = \frac{1}{4} \tag{2.3}$$

Conversely, Suppose $z \in \mathbb{C}$ and $r \in \mathbb{R}$ satisfy (2.3), then $0 \le r \le 1$ and $z = \sqrt{r(1-r)}e^{it}$ for some $t \in \mathbb{R}$. Let $x_1 = \sqrt{1-r}$ and $x_2 = \sqrt{r}e^{it}$. Then $(\overline{x}_1 x_2, |x_2|^2) = (e^{it}\sqrt{r(1-r)}, r) = (z,r)$. Hence,

$$DW(C) = \left\{ (z,r) : |z|^2 + \left(r - \frac{1}{2}\right)^2 = \frac{1}{4} \right\} \subseteq \mathbf{cl}\,(DW(A))$$

and $DW(B) = \mathbf{conv}\{DW(A) \cup DW(C)\}$. So, $\mathbf{cl}\,(DW(A)) = \mathbf{cl}\,(DW(B))$. Furthermore, note that $DW(A)$ contains the interior of $\mathbf{cl}\,(DW(A))$ and $\mathbf{conv}\{(0,0),(1,1),(-1,1)\}$ as subsets because $0, 1, -1 \in \sigma_p(A)$, and that the union of these two subsets of $DW(A)$ contains $DW(C)$. It follows that

$$DW(A) = \mathbf{conv}\{DW(A) \cup DW(C)\} = DW(B).$$

In fact, one easily verifies that $DW(A) = \mathbf{conv}\{(d_n, |d_n|^2) : n \ge 1\}$.

Remarks For an infinite-dimensional operator A, consider the following conditions:

(a) A is normal.

(b) $\mathbf{cl}\,(DW(A)) = \mathbf{conv}\{(\lambda, |\lambda|^2) : \lambda \in \sigma(A)\}$.

(c) $\mathbf{cl}\,(DW(A))$ is a polyhedron.

From our results, we see that for an infinite-dimensional operator A, we have (c) $\Rightarrow$ (a) $\Rightarrow$ (b). By Example 2.5, (b) does not imply (a), and (a) clearly does not imply (c).

Let $D = \operatorname{diag}(1, 1/2, 1/3, 1/4, \dots)$ and $A = D \oplus iD$. Then

$$\mathbf{cl}\,(W(A)) = \mathbf{conv}\{0, 1, i\} = \mathbf{conv}(\sigma(A))$$

is a triangle, and $W(A) = \mathbf{cl}\,(W(A)) \setminus \{0\}$ is not closed. Hence, $\mathbf{cl}\,(W(A))$ being a closed convex polygon does not imply that $W(A)$ is a closed convex polygon even for a normal operator A. In other words, the numerical range analog of (1) and (2) of Theorem 2.3 are not equivalent. Note that our example is also a counter-example of [7, Corollary 1.5-7].

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(Li) Department of Mathematics, College of William & Mary, Williamsburg, VA 23185 (ckli@math.wm.edu).

(Poon) Department of Mathematics, Iowa State University, Ames, IA 50011 (ytpoon@iastate.edu).