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THE DETERMINANT OF THE SUM OF TWO MATRICES

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Let A and B be $n \times n$ matrices over the real or complex field. Lower and upper bounds for $|det(A+B)|$ are given in terms of the singular values of A and B. Extension of our techniques to estimate $|f(A+B)|$ for other scalar-valued functions f on matrices is also considered.

1. INTRODUCTION

We are interested in estimating the determinant of the sum of two square matrices over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ given some partial information about them. For two square matrices A and B, it is well-known that knowing $det(A)$ and $det(B)$ gives no knowledge of $det(A+B)$. For example, if $A = \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$, then $det(A) = det(B) = 0$, but $det(A+B) = z$ (for any $z \in \mathbb{F}$). Although $det(X)$ is the product of the eigenvalues of X, the above example shows that not much can be said about $det(A+B)$ even if the eigenvalues of A and B are known.

Recall that the singular values of X are the nonnegative square roots of the eigenvalues of X^*X ($X^* = X^t$ in the real case). We refer the readers to [**3**, Chapter 3] for the properties and other equivalent characterisations of singular values. It is easy to see that $|det(X)|$ is the product of singular values of X. It turns out that one can obtain a containment region for $det(A+B)$ in terms of the singular values of A and B. We shall present our main theorem and proof in the next section. Extensions of our result and some related problems will be discussed in Section 3.

2. MAIN RESULT AND PROOF

THEOREM 1. *There exist $n \times n$ matrices A and B over $\mathbb{F}$ with singular values $a_1 \geqslant \cdots \geqslant a_n \geqslant 0$ and $b_1 \geqslant \cdots \geqslant b_n \geqslant 0$, respectively, such that $det(A+B) = z \in \mathbb{F}$ if and only if*

$$\prod_{j=1}^{n} (a_j + b_{n-j+1}) \geqslant |z| \geqslant \begin{cases} 0 & \text{if } [a_n, a_1] \cap [b_n, b_1] \neq \emptyset, \\ \left|\prod_{j=1}^{n} (a_j - b_{n-j+1})\right| & \text{otherwise.} \end{cases}$$

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To prove Theorem 1, we need several lemmas and the concept of *weak majorisation*. Recall that for $x, y \in \mathbb{R}^n$, x is weakly majorised by y, denoted by $x \prec^w y$ if the sum of the k smallest entries of x is not smaller than that of y, $k = 1, \ldots, n$.

LEMMA 2. *Suppose A and B have singular values $a_1 \geqslant \cdots \geqslant a_n \geqslant 0$ and $b_1 \geqslant \cdots \geqslant b_n \geqslant 0$, respectively. If $A - B$ has singular values $c_1 \geqslant \cdots \geqslant c_n$, then*

$$(a_1 + b_n, \ldots, a_n + b_1) \prec^w (c_1, \ldots, c_n).$$

Furthermore, if $b_n > a_1$ or $a_n > b_1$, then

$$(c_1, \ldots, c_n) \prec^w (|a_1 - b_n|, \ldots, |a_n - b_1|).$$

PROOF: Note that if X is a square matrix with singular values $s_1 \geqslant \cdots \geqslant s_n$, then the matrix $\begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}$ has eigenvalues $\pm s_1, \ldots, \pm s_n$. Applying the results in [7] to the matrix

$$\begin{pmatrix} 0 & C \\ C^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} - \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix},$$

we see that for any $1 \leqslant i_1 < \cdots < i_k \leqslant n$ and $1 \leqslant j_1 < \cdots < j_k \leqslant n$,

$$\sum_{s=1}^{k} c_{i_s + j_s - s} \leqslant \sum_{s=1}^{k} (a_{i_s} - b_{j_s}).$$

In particular, the sum of the k smallest entries of $(c_1, \ldots, c_n)$ is not larger than that of $(a_1 + b_n, \ldots, a_n + b_1)$. Thus the first assertion follows.

Now suppose $a_n > b_1$. Then $a_1 - b_n \geqslant \cdots \geqslant a_n - b_1 > 0$. Applying the results in [7] to the matrix

$$\begin{pmatrix} 0 & C \\ C^* & 0 \end{pmatrix} + \begin{pmatrix} 0 & -B \\ -B^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix},$$

we see that

$$\sum_{s=1}^{k} c_{n-s+1} - b_s \geqslant \sum_{s=1}^{k} a_{n-s+1}.$$

Thus the sum of the k smallest entries of $((a_1 - b_n), \ldots, (a_n - b_1))$ is not larger than that of $(c_1, \ldots, c_n)$. Similarly, we can show that the sum of the k smallest entries of $((b_1 - a_n), \ldots, (b_n - a_1))$ is not larger than that of $(c_1, \ldots, c_n)$ if $b_n > a_1$. Thus the last assertion of the lemma follows. ∎

LEMMA 3. *Suppose A, B are $n \times n$ matrices which satisfy the hypotheses of Lemma 2. If $a_n > b_1$ or $b_n > a_1$, then $A - B$ is invertible.*

PROOF: Suppose $a_n > b_1$. Then for any unit vector $x \in \mathbb{C}^n$, we have $\|Ax\| \geqslant a_n > b_1 \geqslant \|Bx\|$. As a result, $\|(A - B)x\| \geqslant \|Ax\| - \|Bx\| > 0$ for any unit vector

x, and hence $A+B$ is invertible. Similarly, we can prove that $A+B$ is invertible if $b_n > a_1$. □

Lemma 4. *Suppose $a_1 \geqslant \cdots \geqslant a_n \geqslant 0$ and $b_1 \geqslant \cdots \geqslant b_n \geqslant 0$ are such that $[a_n, a_1] \cap [b_n, b_1] \neq \phi$. There exist real $n \times n$ matrices A, B with the a_i's and b_i's as singular values such that $det(A+B) = 0$.*

Proof: Choose $t \in [a_n, a_1] \cap [b_n, b_1]$. Set $A = \begin{pmatrix} t & 0 \\ \alpha_1 & \alpha_2 \end{pmatrix} \oplus \operatorname{diag}(a_2, \dots, a_{n-1}) \in M_n$, where $\alpha_1, \alpha_2 \in \mathbb{R}$ satisfy $t\alpha_2 = a_1 a_n$ and $t^2 + \alpha_1^2 + \alpha_2^2 = a_1^2 + a_n^2$. Note that the existence of α_1 and α_2 is guaranteed by the assumption that $t \in [a_n, a_1]$. Then A has singular values $a_1 \geqslant \cdots \geqslant a_n$. Similarly, one can construct $B = \begin{pmatrix} -t & 0 \\ \beta_1 & \beta_2 \end{pmatrix} \oplus \operatorname{diag}(b_2, \dots, b_{n-1}) \in M_n$ with singular values $b_1 \geqslant \cdots \geqslant b_n$. It is clear that $det(A+B) = 0$. □

Proof of Theorem 1: ($\Rightarrow$) Suppose A and B have as singular values the a_i's and b_i's, respectively, and suppose $z = det(A+B)$. If $z = 0$, then clearly $|z| \leqslant \prod_{j=1}^{n} (a_j + b_{n-j+1})$. Suppose $A+B$ is nonsingular and has singular values $c_1 \geqslant \cdots \geqslant c_n > 0$. By Lemma 2, $(a_1 + b_n, \dots, a_n + b_1) \prec^w (c_1, \dots, c_n)$. Since the function $f(x) = -\log(x)$ is convex and decreasing for $x > 0$, we have (for example, see [5, Chapter 3, C.1.b]) $-\sum_{i=1}^{n} \log(c_i) \geqslant -\sum_{i=1}^{n} \log(a_i + b_{n-i+1})$. Consequently, $|det(A+B)| = \prod_{i=1}^{n} c_i \leqslant \prod_{i=1}^{n} (a_i + b_{n-i+1})$. Now suppose $[a_n, a_1] \cap [b_n, b_1] = \phi$. Then $(c_1, \dots, c_n) \prec^w (|a_1 - b_n|, \dots, |a_n - b_1|)$. By similar arguments as above, we conclude that $\left|\prod_{i=1}^{n} (a_i - b_{n-i+1})\right| \leqslant \prod_{i=1}^{n} c_i = |det(A+B)|$.

($\Leftarrow$) Let $X = \operatorname{diag}(a_1, \dots, a_n)$ and $Y = \operatorname{diag}(b_n, \dots, b_1)$. Then $det(A+B) = \prod_{i=1}^{n} (a_i + b_{n-i+1})$ if $A = X$ and $B = Y$; $det(A+B) = \prod_{i=1}^{n} (a_i - b_{n-i+1})$ if $A = X$ and $B = -Y$; and $det(A+B) = \prod_{i=1}^{n} (b_i - a_{n-i+1})$ if $A = -X$ and $B = Y$. If $[a_n, a_1] \cap [b_n, b_1] \neq 0$, we can construct suitable A and B such that $det(A+B) = 0$ by Lemma 5. Since the set of real orthogonal matrices with positive determinant is connected, the set

$$S = \{det(U_1 X + U_2 Y) : U_i \text{ is real orthogonal with } det(U_i) = 1, \text{ for } i = 1, 2\}$$

is a line segment. If n is even, then $det(X-Y), det(X+Y) \in S$ and hence $[det(X-Y), det(X+Y)] \subseteq S$. If n is odd, let

$$c = (a_1 + b_n) \prod_{i=2}^{n} (a_i - b_{n-i+1}) \quad \text{and} \quad d = |(a_n - b_1)| \prod_{i=1}^{n-1} (a_i + b_{n-i+1}).$$

Then $c \leqslant d$, $[c, det(X+Y)] \subseteq S$, and $[|det(X-Y)|, d]$ is a subset of the line segment

$$\widetilde{S} = \{det(U_1X + U_2Y) : U_1 \text{ and } U_2 \text{ are real orthogonal with } det(U_1) = \varepsilon = -det(U_2)\},$$

where $\varepsilon = (a_n - b_1)/|a_n - b_1|$. Thus for any $z \in [|det(X-Y)|, det(X+Y)]$, there exist suitable A and B such that $det(A-B) = z$. If $z \leqslant 0$ in the real case, or the argument of z equals $t \neq 0$ in the complex case, where z lies between the upper and lower bounds in Theorem 1, one can first construct suitable A and B so that $det(A+B) = |z|$. Then replace A and B by PA and PB, where $P = \operatorname{diag}(e^{it}, 1, \ldots, 1)$ with $t = -\pi$ when $z < 0$, to get $det(PA + PB) = z$. □

3. Extension and Related Problems

Note that if more about A and B is known, then a better containment region for $det(A+B)$ can be given. For example, by the result in [**2**]:

There exist $n \times n$ complex matrices $A = A^t$ and $B = -B^t$ with singular values $a_1 \geqslant \cdots \geqslant a_n \geqslant 0$ and $b_1 = b_2 \geqslant b_3 = b_4 \geqslant \cdots$ such that $z = det(A+B)$ if and only if

$$det(X-Y) \geqslant |z| \geqslant \begin{cases} 0 & \text{if } [a_n, a_1] \cap [b_n, b_1] \neq \emptyset, \\ |det(\sqrt{-1}X + Y)| & \text{otherwise,} \end{cases}$$

where $X = \sum_{j=1}^{n} a_j E_{jj}$ and $Y = \sum_{k \leqslant (n+1)/2} b_{2k}(E_{2k-1,2k} - E_{2k,2k-1})$.

Here E_{ij} denotes the $n \times n$ matrix with its (i,j) entry equal to one and all other entries equal to zero.

Although our example in Section 1 shows that it is difficult to find a containment region for $det(A+B)$ in terms of the eigenvalues of A and B in general, the situation may be different if A and B are normal. In fact, Marcus [4] and Oliveira [6] independently conjectured that:

If A and B are $n \times n$ complex normal matrices with eigenvalues $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$, respectively, then $det(A+B)$ lies in the convex hull of the points of the form $\prod_{i=1}^{n} (\alpha_i + \beta_{\sigma(i)})$, where σ is a permutation of the set $\{1, \ldots, n\}$.

A number of special cases of this conjecture have been verified, but the general problem remains open (for example, see [**1**]).

It is worthwhile to point out that one can actually deduce the following result from our proof.

Theorem 5. *Suppose $f(x_1, \ldots, x_n)$ is a Schur concave function on vectors with nonnegative entries, and is increasing in each coordinate. For $X \in M_n$, denote by $f(X)$ the functional value of f on the singular values of X. If A and B have singular* *values* $a_1 \geqslant \cdots \geqslant a_n$ *and* $b_1 \geqslant \cdots \geqslant b_n$*, then* $f(a_1 + b_n, \ldots, a_n + b_1) \geqslant f(A + B)$*. If, in addition,* $[a_n, a_1] \cap [b_n, b_1] = \phi$*, then* $f(A + B) \geqslant f(|a_1 - b_n|, \ldots, |a_n - b_1|)$.

The kth elementary symmetric function, $1 \leqslant k \leqslant n$, is an example of a Schur concave function that is increasing in each coordinate. Of course, it reduces to $det(X)$ when $k = n$. It would be interesting to have a lower bound for $f(A + B)$ in general.

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