INTERPOLATION BY COMPLETELY POSITIVE MAPS

CHI-KWONG LI AND YIU-TUNG POON

Dedicated to Professor John Conway on the occasion of his retirement.

ABSTRACT. Given commuting families of Hermitian matrices $\{A_1, \dots, A_k\}$ and $\{B_1, \dots, B_k\}$, conditions for the existence of a completely positive map Φ, such that $\Phi(A_j) = B_j$ for $j = 1, \dots, k$, are studied. Additional properties such as unital or / and trace preserving on the map Φ are also considered. Connections of the study to dilation theory, matrix inequalities, unitary orbits, and quantum information science are mentioned.

1. INTRODUCTION

Denote by $M_{n,m}$ the set of $n \times m$ complex matrices, and use M_n to denote $M_{n,n}$. Let H_n be the set of Hermitian matrices in M_n. A matrix $A \in H_n$ is positive semidefinite if all eigenvalues of A are nonnegative. A linear map $\Phi : M_n \to M_m$ is *positive* if it maps positive semidefinite matrices to positive semidefinite matrices. Suppose $M_k(M_n)$ is the algebra of block matrices of the form $(A_{ij})_{1\le i,j\le k}$ with $A_{ij} \in M_n$ for each pair of (i, j). A linear map $\Phi : M_n \to M_m$ is *completely positive* if for each positive integer k, the map $I_k \otimes \Phi : M_k(M_n) \to M_k(M_m)$ defined by $(I_k \otimes \Phi)(A_{ij}) = (\Phi(A_{ij}))$ is positive.

The purpose of this paper is to study the following.

Problem 1.1. *Given $A_1, \dots, A_k \in M_n$ and $B_1, \dots, B_k \in M_m$, determine the necessary and sufficient condition for the existence of a completely positive map $\Phi : M_n \to M_m$ such that $\Phi(A_j) = B_j$ for $j = 1, \dots, k$, and possibly with the additional properties that $\Phi(I_n) = I_m$ or/and Φ is trace preserving.*

Clearly, this can be viewed as an interpolation problem by completely positive maps. Denote by H_n the set of Hermitian matrices in M_n. Since for a positive linear map Φ satisfying $\Phi(X) = Y$ if and only if $\Phi(X^*) = Y^*$ for any $(X, Y) \in M_n \times M_m$, we can focus on the study of Problem 1.1 for $\{A_1, \dots, A_k\} \subseteq H_n$ and $\{B_1, \dots, B_k\} \subseteq H_m$

Over half a century ago, Steinspring [19] introduced completely positive maps in the study of dilation problems for operators. Since then, the area has been studied extensively [15, 16]. In particular, researchers have obtained interesting structure theorem for completely positive maps on matrices. For example, Choi [2] (see also [10]) showed that a linear map $\Phi : M_n \to M_m$ is

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completely positive if and only if there exist $F_1, \ldots, F_r \in M_{n,m}$ such that

$$\Phi(A) = \sum_{j=1}^{r} F_j^* A F_j. \tag{1.1}$$

This is called an *operator sum representation* of the completely positive map Φ.

In the context of quantum information theory, every quantum operation is a completely positive map sending quantum states to quantum states, where quantum states are represented as *density matrices*, i.e., positive semidefinite matrices with trace one. Because a completely positive map $\Phi : M_n \to M_m$ representing a quantum operation will send density matrices to density matrices, the map Φ is *trace preserving*, i.e., $\Phi(A) \in M_m$ and $A \in M_n$ always have the same trace. Therefore, in quantum information science, most studies are on trace preserving completely positive maps. On the other hand, in the C^*-algebra context, since the trace function may not be defined, most studies are on *unital* maps, i.e., $\Phi(I_n) = I_m$. While rich theory has been developed for completely positive maps, for example, see [15, 16], there are many basic problems motivated by applied topics which deserve further study.

In quantum information science, one has to study and construct quantum operations sending a specific family of density operators to another family. This clearly reduces to Problem 1.1 if we restrict our attention to density matrices $A_1, \ldots, A_k \in M_n$ and $B_1, \ldots, B_k \in M_m$; see [14].

Using the operator sum representation (1.1) of completely positive maps, and the inner product $\langle A, B\rangle = \operatorname{tr}(AB^*)$ for $A, B \in M_{m,n}$, one has the following result showing the connection between trace preserving completely positive maps and unital completely positive maps.

Proposition 1.2. *Suppose $\Phi : M_n \to M_m$ is a completely positive map with operator sum representation in (1.1). Then Φ is unital if and only if $\sum_{j=1}^r F_j^* F_j = I_m$; Φ is trace preserving if and only if $\sum_{i=1}^r F_j F_j^* = I_n$. Moreover, the dual linear map $\Phi^* : M_m \to M_n$ defined by*

$$\Phi^*(B) = \sum_{j=1}^{r} F_j B F_j^*$$

is the unique linear map satisfying $\langle \Phi(A), B\rangle = \langle A, \Phi^*(B)\rangle$ for all $(A, B) \in M_n \times M_m$. Consequently, Φ is unital if and only if Φ^* is trace preserving.

The above proposition provides a link between problems and results for trace preserving completely positive maps and unital completely positive maps. Therefore, one might expect that the results and proofs for the two types of problems can be converted to each other easily. However, this does not seem to be the case as shown in our results. In fact, some results in trace preserving completely positive maps are more involved in our study, and they have no analogs for unital completely positive maps; see Remark 2.3 b), and the remarks after Corollary 3.4 and Theorem 3.6.

It is known that the study of completely positive maps are closely related to the dilations of operators. Recall that a matrix $B \in M_m$ has a *dilation* $A \in M_n$ if there is an $n \times m$ matrix V

such that $V^*V = I_m$ and $V^*AV = B$. The next result shows that Problem 1.1 can be formulated as problems involving dilations and principal submatrices of a matrix.

Proposition 1.3. *Suppose $\Phi : M_n \to M_m$ is a completely positive map with operator sum representation* (1.1). *If $F = \begin{bmatrix} F_1 \\ \vdots \\ F_r \end{bmatrix}$, then $\Phi(A) = F^*(I_r \otimes A)F$. If $\tilde{F} = [F_1 \ \cdots \ F_r]$ and $\tilde{F}^* A \tilde{F} = (A_{ij})$ with $A_{11}, \ldots, A_{rr} \in M_m$, then $\Phi(A) = A_{11} + \cdots + A_{rr}$. Furthermore, the following hold.*

(a) *The map Φ is unital if and only if $F^*F = I_m$.*

(b) *The map Φ is trace preserving if and only if $\tilde{F}\tilde{F}^* = I_n$.*

Proposition 1.4. *Suppose $\Phi : M_n \to M_m$ is a completely positive map. Given unitaries $U \in M_n$ and $V \in M_m$, define $\Psi : M_n \to M_m$ by $\Psi(X) = V^*\Phi(U^*XU)V$. Then Ψ is also completely positive. Furthermore, Ψ is unital and/or trace preserving if and only if Φ has the corresponding property. If $\Phi(X) = \sum_{j=1}^r F_j^* X F_j$, then $\Psi(X) = \sum_{j=1}^r (UF_jV)^* X (UF_jV)$.*

Propositions 1.2 – 1.4 will be used in the subsequent discussion. Our paper is organized as follows. In Section 2, we determine the condition for the existence of completely positive maps (possibly with additional conditions such as unital or/and trace presering) $\Phi : M_n \to M_m$ sending a given commuting family of matrices in H_n to another one in H_m. In Section 3, we give a more detailed analysis for the case when each family has only one matrix. Some related results, additional remarks, and open problems will be mentioned in Section 4.

2. Completely positive maps between commuting families

In this section, we consider (unital) completely positive maps $\Phi : M_n \to M_m$ sending a given commuting family of matrices in H_n to another commuting family in H_m.

Theorem 2.1. *Let $\{A_1, \ldots, A_k\} \subseteq H_n$ and $\{B_1, \ldots, B_k\} \subseteq H_m$ be two commuting families. Then there exist unitary matrices $U \in M_n$ and $V \in M_m$ such that U^*A_iU and VB_iV^* are diagonal matrices with diagonals $\mathbf{a}_i = (a_{i1}, \ldots, a_{in})$ and $\mathbf{b}_i = (b_{i1}, \ldots, b_{im})$ respectively, for $i = 1, \ldots, k$. The following conditions are equivalent.*

(a) *There is a completely positive map $\Phi : M_n \to M_m$ such that $\Phi(A_i) = B_i$ for $i = 1, \ldots, k$.*

(b) *There is an $n \times m$ nonnegative matrix $D = (d_{pq})$ such that*

$$(b_{ij}) = (a_{ij})D.$$

Suppose (b) *holds. For $1 \le j \le m$, let F_j be the $n \times n$ matrix having the jth column equal to $(\sqrt{d_{1j}}, \sqrt{d_{2j}}, \ldots, \sqrt{d_{nj}})^t$ and zero elsewhere. Then we have*

$$B_i = \sum_{j=1}^r (UF_jV)^* A_i (UF_jV), \qquad i = 1, \ldots, k, \tag{2.1}$$

Furthermore,

(1) *Φ in* (a) *is unital if and only if D in* (b) *can be chosen to be column stochastic.*

(2) Φ *in* (a) *is trace preserving if and only if* D *in* (b) *can be chosen to be row stochastic.*
(3) Φ *in* (a) *is unital and trace preserving if and only if* D *in* (b) *can be chosen to be doubly stochastic.*

Proof. By Proposition 1.4, we may assume that $A_i = \operatorname{diag}(\mathbf{a}_i)$ and $B_i = \operatorname{diag}(\mathbf{b}_i)$ for $i = 1, \ldots, k$, and take $U = I_n$, $V = I_m$ in (2.1).

(a) $\Rightarrow$ (b): Suppose there is a completely positive map $\Phi : M_n \to M_m$ such that $\Phi(A_i) = B_i$ for $i = 1, \ldots, k$. By (1.1), we have $F^j = \left(f^j_{pq}\right) \in M_{n,m}$, $j = 1, \ldots, r$, such that

$$\Phi(X) = \sum_{j=1}^{r} F_j^* X F_j.$$

For $1 \le p \le n$ and $1 \le q \le m$, let $d_{pq} = \sum_{j=1}^{r} |f^j_{pq}|^2$. Then $D = (d_{pq})$ is an $n \times m$ nonnegative matrix such that

$$(b_{ij}) = (a_{ij})D.$$

(b) $\Rightarrow$ (a): Suppose $D = (d_{pq})$ is a nonnegative matrix satisfying $(b_{ij}) = (a_{ij})D$. Let F_j be defined as in the theorem. Then direct computation shows that for every $\mathbf{x} = (x_1, \ldots, x_n)$, $\sum_{j=1}^{r} F_j^* \operatorname{diag}(\mathbf{x}) F_j$ is a diagonal matrix with diagonal $\mathbf{y} = \mathbf{x}D$. Hence, for $i = 1, \ldots, k$, we have

$$\operatorname{diag}(\mathbf{b}_i) = \sum_{j=1}^{m} F_j^* \operatorname{diag}(\mathbf{a}_i) F_j,$$

for $i = 1, \ldots, k$.

For $N \ge 1$, let $\mathbf{1}_N$ be a row vector of N 1's, then we have

$$\Phi \text{ is unital } \Leftrightarrow \Phi(I_n) = I_m \Leftrightarrow \mathbf{1}_m = \mathbf{1}_n D \Leftrightarrow D \text{ is column stochastic.}$$

This proves (1). The proof for cases (2) and (3) are similar. □

Condition (b) in Theorem 2.1 has been studied in [5, 17, 1] with row (column, doubly, respectively) D. An analog of (3) for II_1 factor is also proven in [1].

A completely positive map $\Phi : M_n \to M_n$ is called *mixed unitary* if there exist unitary matrices $U_1, \ldots, U_r \in M_n$ and positive numbers $t_1, \ldots, t_r$ summing up to 1 such that

$$\Phi(X) = \sum_{j=1}^{r} t_j U_j^* X U_j.$$

Clearly, every mixed unitary completely positive map is unital and trace preserving. For $n \ge 3$, there exists [11] a unital trace preserving completely positive map which is not mixed unitary.

By the Birkhoff Theorem [13], a doubly stochastic matrix D can be expressed in the form

$$D = \sum_{j=1}^{r} t_j P_j \tag{2.2}$$

for some positive numbers $t_1, \ldots, t_r$ summing up to 1 and permutation matrices $P_1, \ldots, P_r \in M_n$. Using this result and (3) in Theorem 2.1, we have the following corollary.

Corollary 2.2. *Under the hypothesis of Theorem 2.1, suppose there is a unital trace preserving completely positive map* Φ *satisfying* (a)*, and* D *is a doubly stochastic matrix satisfying* (b)*. Let* D *be expressed as in (2.2). Then the mixed unitary map* $\Psi : M_n \to M_n$ *defined by*

$$\Psi(X) = \sum_{j=1}^{r} t_j (UP_jV)^* X (UP_jV)$$

also satisfies $\Psi(A_i) = B_i$ *for* $i = 1, \ldots, k$.

Remark 2.3. The following remarks concerning Theorem 2.1 are in order.

a) In Theorem 2.1, if the matrices U and V in the hypothesis are merely invertible instead of unitary, conditions (a) and (b) are still equivalent. In other words, the result applies to two families $\{A_1, \ldots, A_k\} \subseteq H_n$ and $\{B_1, \ldots, B_k\} \subseteq H_m$ such that each family is simultaneously congruent to diagonal matrices.

b) To check conditions (b) and the corresponding ones in (1) , (2) and (3), one can use standard linear programming techniques.

   To check condition (1), one can divide the problem of finding a column stochastic D into m independent problems of determining the columns $\mathbf{d}_1, \ldots, \mathbf{d}_m$ of D as follows.

   Set $A = (a_{ij})_{1 \le i \le k, 1 \le j \le n}$. For each $q = 1, \ldots, m$, determine a nonnegative vector $\mathbf{d}_q \in \mathbb{R}^n$ with entries summing up to one such that $A\mathbf{d}_q = (b_{1q}, \ldots, b_{kq})^t$.

   Clearly, the solution set of each of the problems is a convex polyhedron, namely,

$$P_q = \left\{ \mathbf{d} = \begin{bmatrix} d_{1q} \\ \vdots \\ d_{nq} \end{bmatrix} : d_{pq} \ge 0, \ \sum_{p=1}^{n} d_{pq} = 1, \quad \text{and } A\mathbf{d} = \begin{bmatrix} b_{1q} \\ \vdots \\ b_{kq} \end{bmatrix} \right\}$$

   in $\mathbb{R}^n$. By standard convex analysis, if A has rank k, then the extreme points of the polyhedron P_q, if non-empty, has at most $k+1$ nonzero entries. Thus, one can construct a sparse matrix D as a solution using the extreme points in the solution sets P_q.

   To check condition (2), one can construct an extreme point (with sparse pattern) of the set of row stochastic matrices D satisfying $(b_{ij}) = (a_{ij})D$. However, unlike the checking of (1), one cannot treat individual rows separately to reduce the complexity of the computation.

c) In the construction of $F_1, \ldots, F_r$ in the last assertion of the theorem, we see that each F_j has rank at most $\ell = \min\{m, n\}$ so that the corresponding completely positive map Φ is a super ℓ-positive map [18].

d) Note that in our construction, if $A_1, \ldots, A_k$ and $B_1, \ldots, B_k$ are in diagonal form, then the map $A_i \mapsto F_j^* A_i F_j$ has only one nonzero entry at the (j, j) position obtained by taking a nonnegative combination of the diagonal entries of A_i. In the context of quantum information science, it is easy to implement the map (quantum operation) Φ on diagonal matrices because all the actions only take place at the diagonal entries (classical channels).

3. Completely positive maps on a single matrix

For a single matrix, we can give a more detailed analysis of the result in Theorem 2.1, and show that the study is related to other topics such as eigenvalue inequalities. Moreover, the results show that there are some results on trace preserving completely positive maps with no analogs for unital completely positive maps, and vice versa.

3.1. Unital completely positive maps.

Theorem 3.1. *Suppose $A \in H_n$ and $B \in H_m$ have eigenvalues $a_1, \dots, a_n$ and $b_1, \dots, b_m$, respectively. Let $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_m)$. The following conditions are equivalent.*

(a) *There is a (unital) completely positive map $\Phi : M_n \to M_m$ such that $\Phi(A) = B$.*

(b) *There is a nonnegative (column stochastic) matrix $D = (d_{pq})$ such that $\mathbf{b} = \mathbf{a}D$.*

(c) *There are real numbers $\gamma_1, \gamma_2 \geq 0$ (with $\gamma_1 = \gamma_2 = 1$) such that*

$$\gamma_2 \min\{a_i : 1 \leq i \leq n\} \leq b_j \leq \gamma_1 \max\{a_i : 1 \leq i \leq n\}. \tag{3.1}$$

for all $1 \leq j \leq m$.

Proof. By Theorem 2.1, (a) and (b) are equivalent. Without loss of generality, we may assume that $a_1 \geq \cdots \geq a_n$ and $b_1 \geq \cdots \geq b_m$.

(b) $\Rightarrow$ (c): Suppose there is a nonnegative matrix $D = (d_{pq})$ such that $\mathbf{b} = \mathbf{a}D$. Let $\gamma_1 = \sum_{p=1}^n d_{p1}$ and $\gamma_2 = \sum_{p=1}^n d_{pm}$. Then for each $1 \leq j \leq m$, we have

$$\gamma_2 a_n = \sum_{p=1}^n d_{pm} a_n \leq \sum_{p=1}^n d_{pm} a_p = b_m \leq b_j \leq b_1 = \sum_{p=1}^n d_{p1} a_p \leq \sum_{p=1}^n d_{p1} a_1 = \gamma_1 a_1.$$

If D is column stochastic, then it follows from definition that $\gamma_1 = \gamma_2 = 1$.

(c) $\Rightarrow$ (b): Suppose (c) holds. Then for each $i = 1, \dots m$, there exists $0 \leq t_i \leq 1$ such that $b_i = t_i \gamma_1 a_1 + (1 - t_i)\gamma_2 a_n$. Let D be the $n \times m$ matrix

$$\begin{bmatrix} t_1\gamma_1 & t_2\gamma_1 & \cdots & t_m\gamma_1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \\ (1-t_1)\gamma_2 & (1-t_2)\gamma_2 & \cdots & (1-t_m)\gamma_2 \end{bmatrix}$$

Then $\mathbf{b} = \mathbf{a}D$ and D is column stochastic if $\gamma_1 = \gamma_2 = 1$. □

Corollary 3.2. *If $A \in H_n$ and $B \in H_m$ are nonzero positive semi-definite, then there is a completely positive map $\Phi : M_n \to M_m$ such that $\Phi(A) = B$, and there is a completely positive map $\Psi : M_m \to M_n$ such that $\Psi(B) = A$.*

Using Theorem 3.1, one can construct a pair of density matrices $(A, B) \in H_n \times H_m$ such that there is a unital completely positive map Φ such that $\Phi(A) = B$, but there is no unital completely positive map Ψ such that $\Psi(B) = A$.

3.2. Trace preserving completely positive maps.

Theorem 3.3. *Suppose $A \in H_n$ and $B \in H_m$ have eigenvalues $a_1, \dots, a_n$ and $b_1, \dots, b_m$ respectively. Let $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_m)$. The following conditions are equivalent.*

(a) *There is a trace preserving completely positive map $\Phi : M_n \to M_m$ such that $\Phi(A) = B$.*

(b) *There exists an $n \times m$ row stochastic matrix D such that $\mathbf{b} = \mathbf{a}D$. Moreover, we can assume that the* pth *and* qth *row are identical whenever $a_p a_q > 0$, and the* pth *row of D can be arbitrary (nonnegative with entries summing up to 1) if $a_p = 0$.*

(c) *We have $\operatorname{tr} A = \operatorname{tr} B$ and $\sum_{p=1}^n |a_p| \geq \sum_{q=1}^m |b_q|$. Equivalently, $\operatorname{tr} A = \operatorname{tr} B$ and the sum of the positive (negative) eigenvalues of A is not smaller (not larger) than the sum of the positive (negative) eigenvalues of B.*

Proof. For simplicity, we assume that $a_1 \geq \cdots a_r \geq 0 > a_{r+1} \geq \cdots \geq a_n$ and $b_1 \geq \cdots \geq b_s \geq 0 > b_{s+1} \geq \cdots \geq b_m$. Let $a_+ = \sum_{p=1}^r a_p$, $a_- = \sum_{p=r+1}^n a_p$ and $b_+ = \sum_{q=1}^s b_q$, $b_- = \sum_{q=s+1}^m b_q$.

By Theorem 2.1, we have (b) $\Rightarrow$ (a). Also, by Theorem 2.1, if (a) holds, then $\mathbf{b} = \mathbf{a}D$ for an $n \times m$ row stochastic matrix D. Next, we show that D can be chosen to satisfy the second assertion of condition (b). To this end, let $\mathbf{d}_1, \dots, \mathbf{d}_n$ be the rows of D. Clearly, if $a_p = 0$, we can replace the pth row of D by any nonnegative vectors with entries summing up to 1 to get $\tilde{D}$ and we still have $\mathbf{a}\tilde{D} = \mathbf{a}D = \mathbf{b}$. Now, suppose $a_1 > 0$. Then $a_+ > 0$. We can replace the first r rows (or the rows correspond to $a_p > 0$) by

$$\mathbf{d}_+ = \sum_{j=1}^r \frac{a_j}{a_+} \mathbf{d}_j$$

to obtain $\tilde{D}$. Then $\mathbf{d}_+$ has nonnegative entries summing up to 1, and $\mathbf{a}\tilde{D} = \mathbf{a}D = \mathbf{b}$. Similarly, suppose $a_n < 0$. Then $a_- < 0$. We can further replace the last $n - s$ row of D by

$$\mathbf{d}_- = \sum_{j=s+1}^n \frac{a_j}{a_-} \mathbf{d}_j$$

to obtain $\tilde{D}$. Then $\mathbf{d}_-$ has nonzero entries summing up to 1, and $\mathbf{a}\tilde{D} = \mathbf{a}D = \mathbf{b}$.

(b) $\Rightarrow$ (c): Suppose $\mathbf{b} = \mathbf{a}D$. We have

$$\operatorname{tr} B = \sum_{q=1}^m b_q = \sum_{q=1}^m \sum_{p=1}^n a_p d_{pq} = \sum_{p=1}^n a_p \left(\sum_{q=1}^m d_{pq} \right) = \sum_{p=1}^n a_p = \operatorname{tr} A$$

and

$$\sum_{q=1}^m |b_q| = \sum_{q=1}^m |\sum_{p=1}^n a_p d_{pq}| \leq \sum_{q=1}^m \sum_{p=1}^n |a_p| d_{pq} = \sum_{p=1}^n |a_p| \left(\sum_{q=1}^m d_{pq} \right) = \sum_{p=1}^n |a_p|\,.$$

Since

$$\sum_{i=1}^n a_i = \sum_{j=1}^m b_j, \quad \sum_{i=1}^n |a_i| = a_+ - a_- = \sum_{i=1}^n a_i - 2a_- = 2a_+ - \sum_{i=1}^n a_i,$$

and

$$\sum_{j=1}^{m}|b_j| = b_+ - b_- = \sum_{j=1}^{m} b_j - 2b_- = 2b_+ - \sum_{j=1}^{m} b_j,$$

the last assertion of (c) follows.

(c) $\Rightarrow$ (b): Suppose $\operatorname{tr} A = \operatorname{tr} B$, $a_+ \geq b_+$ and $a_- \leq b_-$. Let

$$t_q = \begin{cases} \dfrac{b_q}{a_+} & \text{for } 1 \leq q \leq s, \\[2ex] \dfrac{b_q}{a_-} & \text{for } s < q \leq m. \end{cases}$$

Here, if $a_+ = 0$ then $b_+ = 0$, and we can set $t_q = 0$ for $1 \leq q \leq s$. If $a_- = 0$ then $b_- = 0$, and $s = m$. Therefore, $t_q \geq 0$ for all $1 \leq q \leq m$. We have

$$a_+ \geq b_+ = (a_+)\sum_{q=1}^{s} t_q, \qquad \text{and} \qquad |a_-| \geq |b_-| = |a_-| \sum_{q=s+1}^{n} t_q.$$

Let $u = 1 - \sum_{q=1}^{s} t_q \geq 0$, $v = 1 - \sum_{q=s+1}^{n} t_q \geq 0$ and D be an $n \times m$ row stochastic matrix with

$$p\,\text{th row} = \begin{cases} (t_1, t_2, \ldots, t_s, 0, \ldots, 0, u) & \text{for } 1 \leq p \leq r, \\[1ex] (0, \ldots, 0, t_{s+1}, t_{s+2}, \ldots, t_{m-1}, t_m + v) & \text{for } r+1 \leq p \leq n\,. \end{cases}$$

Since

$$ua_+ + va_- = (a_+ - b_+) + (a_- - b_-) = 0,$$

we have $\mathbf{b} = \mathbf{a}D$. □

Corollary 3.4. *If $A \in H_n$ and $B \in H_m$ are density matrices, i.e. positive semi-definite and $\operatorname{tr} A = \operatorname{tr} B = 1$, then there is a completely positive map $\Phi : M_n \to M_m$ such that $\Phi(A) = B$, and there is a completely positive map $\Psi : M_m \to M_n$ such that $\Psi(B) = A$.*

As remarked after Corollary 3.2, one may not be able to find a unital completely positive map taking a density matrix $B \in H_m$ to another density matrix $A \in H_n$ even if there is a unital positive completely positive map sending A to B.

3.3. Unital trace preserving completely positive maps.

Suppose there is a unital completely positive map sending A to B, and also a trace preserving completely positive map sending A to B. Is there a unital trace preserving completely positive map sending A to B? If the answer were "yes", then one can reduce a complicated problem into two relatively simple problems. Unfortunately, the following example shows that the answer is negative.

Example 3.5. Suppose $A = \operatorname{diag}(4, 1, 1, 0)$ and $B = \operatorname{diag}(3, 3, 0, 0)$. By Theorems 3.1 and 3.3 there is a trace preserving completely positive map sending A to B, and also a unital completely positive map sending A to B. Let $A_1 = A - I_4 = \operatorname{diag}(3, 0, 0, -1)$ and $B_1 =$

$B - I_4 = \mathrm{diag}\,(2, 2, -1, -1)$. By Theorem 3.3, there is no trace preserving completely positive map sending A_1 to B_1. Hence, there is no unital trace preserving completely positive map sending A to B.

As shown in Corollary 2.2, if there is a unital trace preserving map sending a commuting family in H_n to a commuting family in H_m, then we may chose the map to be mixed unitary. In the following, we show that for the case when $k = 1$ in Theorem 2.1, one can even assume that the map is the average of n unitary similarity transforms.

Let $\mathbf{a} = (a_1, \ldots, a_n), \mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n$. We say that $\mathbf{b}$ is majorized by $\mathbf{a}$ ($\mathbf{b} \prec \mathbf{a}$) if for every $1 \le k < n$, the sum of the k largest entries of $\mathbf{b}$ is less than or equal to the sum of the k largest entries of $\mathbf{a}$, and $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$.

Theorem 3.6. *Suppose $A \in H_n$ and $B \in H_m$ have eigenvalues $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ respectively. Let $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_m)$. The following are equivalent.*

(a) *There exists a unital trace preserving completely positive map Φ such that $\Phi(A) = B$.*

(a1) *There exists a mixed unitary completely positive map Φ such that $\Phi(A) = B$.*

(a2) *There exist unitary matrices $U_1, \ldots, U_n \in M_n$ such that $B = \frac{1}{n}\sum_{j=1}^n U_j^* A U_j$.*

(a3) *For each $t \in \mathbb{R}$, there exists a trace preserving completely positive map Φ_t such that $\Phi_t(A - tI) = B - tI$.*

(b) *There is a doubly stochastic matrix D such that $\mathbf{b} = \mathbf{a}D$.*

(b1) *There is a unitary matrix $W \in M_n$ such that $W\mathrm{diag}\,(a_1, \ldots, a_n)W^*$ has diagonal entries $b_1, \ldots, b_n$.*

(c) $\mathbf{b} \prec \mathbf{a}$.

Moreover, if condition (b) holds and $D = \sum_{\ell=1}^r t_\ell P_\ell$ such that $t_1, \ldots, t_r$ are positive numbers summing up to 1 and $P_1, \ldots, P_r$ are permutation matrices, then $B = \sum_{j=1}^r V^* P_j^t U^* A U P_j V$, where $U^*AU = \mathrm{diag}\,(a_1, \ldots, a_n)$ and $V^*BV = \mathrm{diag}\,(b_1, \ldots, b_n)$.

Proof. (b) $\iff$ (c) is a standard result of majorization; see [13].

The implications (a2) $\Rightarrow$ (a1) $\Rightarrow$ (a) $\Rightarrow$ (a3) are obvious.

(a3) $\Rightarrow$ (c) : We may assume that $a_1 \ge \cdots \ge a_n$ and $b_1 \ge \cdots \ge b_n$. For $1 \le k < n$ choose t such that $a_k \ge t \ge a_{k+1}$. Then there is a trace preserving completely positive linear map Φ_t such that $\Phi_t(A - tI_n) = B - tI_n$. By Theorem 3.3, the sum of the k positive eigenvalues of $B - tI_n$ is no larger than that of $A - tI_n$. Thus,

$$\sum_{i=1}^k b_i - kt = \sum_{i=1}^k (b_i - t) \le \sum_{i=1}^k (a_i - t) = \sum_{i=1}^k a_i - kt\,. \tag{3.2}$$

We see that $\sum_{i=1}^k b_i \le \sum_{i=1}^k a_i$ for $k = 1, \ldots, n-1$. Since $\mathrm{tr}\,A = \mathrm{tr}\,B$, we have $\mathbf{b} \prec \mathbf{a}$.

(c) $\Rightarrow$ (b1) is a result of Horn [8].

(b1) $\Rightarrow$ (a2) : Suppose W is a unitary matrix such that the diagonal of $W\mathrm{diag}\,(a_1,\dots,a_n)W^*$ has diagonal entries $b_1,\dots,b_n$. Let $w = e^{i2\pi/n}$, $P = \mathrm{diag}\,(1,w,\dots,w^{n-1})$. Then

$$B = \frac{1}{n}\sum_{j=1}^{n} UW(P^j)^*WUAU^*W^*(P^j)U^*.$$

Thus, (a2) holds. $\square$

In the context of quantum information theory, a completely positive map in condition (a1) of the above theorem is a *mixed unitary quantum channel/operation.* By the above theorem, the existence of a mixed unitary quantum channel Φ taking a quantum state $A \in H_n$ to a quantum state $B \in H_m$ can be described in terms of trace preserving completely positive maps, namely, condition (a3). However, despite the duality of the two classes of maps, there is no analogous condition in terms of unital completely positive map; see Proposition 1.2.

4. Additional remarks and future research

To study unital completely positive maps connecting two families $\{A_1,\dots,A_k\} \subseteq H_n$ and $\{B_1,\dots,B_k\} \subseteq H_m$, one can use the results on completely positive maps and add I_n and I_m to the two families.

The following result shows that the study of a completely positive maps sending $A_1,\dots,A_k \in H_n$ to $B_1,\dots,B_k \in H_m$ can be reduced to the study of unital completely positive maps.

Theorem 4.1. *Let $A_1,\dots,A_k \in H_n$, and $B_1,\dots,B_k \in H_m$. There is a completely positive map $\Phi : M_n \to M_m$ such that $\Phi(A_j) = B_j$ for $j = 1,\dots,k$ if and only if there exists $\gamma > 0$ and a unital completely positive map $\Psi : M_{n+1} \to M_m$ such that $\Psi(A_j \oplus [0]) = \gamma^{-1}B_j$ for $j = 1,\dots,k$.*

Proof. Suppose $\Phi : M_n \to M_m$ has operator sum representation (1.1) and satisfies $\Phi(A_j) = B_j$ for $j = 1,\dots,k$. Let $\Phi(I_n) = P \in H_m$. Choose $\gamma > 0$ such that $\gamma I_m - P$ is positive semi-definite. Then we have $\gamma I_m - P = \sum_{j=1}^{s} g_j^* g_j$ for some $1 \times m$ matrices g_j. For $j = 1,\dots,\tilde{r}$ with $\tilde{r} = \max\{r,s\}$, let $\tilde{F}_j \in M_{n+1,m}$ be such that $\tilde{F}_j = \begin{bmatrix} F_j \\ g_j \end{bmatrix}$, where $F_j = 0$ if $j > r$ and $g_j = 0$ if $j > s$. Define $\Psi : M_{n+1} \to M_m$ by $\Psi(X) = \dfrac{1}{\gamma}\displaystyle\sum_{j=1}^{\tilde{r}} \tilde{F}_j^* X \tilde{F}_j$. One readily checks that $\Psi(I_{n+1}) = I_m$ and $\Psi(A_j \oplus [0]) = \gamma^{-1}B_j$ for $j = 1,\dots,k$.

Conversely, suppose $\gamma > 0$ and $\Psi : M_{n+1} \to M_m$ is a unital completely positive map such that $\Psi(A_j \oplus [0]) = \gamma^{-1}B_j$ for $j = 1,\dots,k$, then one can check $\Phi : M_n \to M_m$ defined by $\Phi(X) = \gamma\Psi(X \oplus [0])$ is a completely positive map satisfying $\Phi(A_j) = B_j$ for $j = 1,\dots,k$. $\square$

Finding a unital completely positive map connecting two general (non-commuting) families of Hermitian matrices is very challenging. In the case of non-commuting families $\{A_1, A_2\} \subset H_n$, and $\{B_1, B_2\} \subseteq H_m$ with two elements, the problems reduce to the study of unital completely positive maps Φ satisfying $\Phi(A_1 + iA_2) = B_1 + iB_2$. For $n = 2, 3$. There are partial answers of the problem in terms of the *numerical range* and *dilation* of operators. Recall that the numerical range of $T \in M_n$ is the set

$$W(T) = \{x^*Tx : x \in \mathbb{C}^n,\ x^*x = 1\},$$

and T has a *dilation* $\tilde{T} \in M_m$ if there is an $n \times m$ matrix X such that $XX^* = I_n$ and $X\tilde{T}X^* = T$. We have the following result; see [3, 4].

Theorem 4.2. *Let $(A, B) \in M_n \times M_m$. Suppose $n = 2$, or $n = 3$ such that A is unitarily reducible, i.e., A is untiarily similar to $A_1 \oplus [\alpha]$ for some $A_1 \in M_2$ and $\alpha \in \mathbb{C}$. Then the following conditions are equivalent.*

(a) *There is a unital completely positive map $\Phi : M_n \to M_m$ such that $\Phi(A) = B$.*
(b) *B has a dilation of the form $I_r \otimes A$.*
(c) *$W(B) \subseteq W(A)$.*

Special cases of the above theorem include the case when $A \in M_3$ is a normal matrix. However, there are examples showing that the result fails if A is an arbitrary matrix in M_3 or an arbitrary normal matrix in M_4; see [3].

In connection to Theorem 4.2, one may ask for the condition of $A \in M_n$ to be a dilation of $B \in M_m$ itself. The problem is challenging even for normal matrices A and B; see [9].

Also, it is interesting to impose condition on the Kraus (Choi) rank, i.e., the minimum number of matrices $F_1, \ldots, F_r$ needed in the operator sum representation of the completely positive maps. As mentioned in Proposition 1.3, the study is related to the study of principal submatrices of a Hermitian matrices, which is related to the study of spectral inequalities and Littlewood-Richardson rule; see [6, 12].

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Department of Mathematics, College of William & Mary, Williamsburg, VA 23187
E-mail address: ckli@math.wm.edu

Department of Mathematics, Iowa State University, Ames, IA 50051
E-mail address: ytpoon@iastate.edu