The Ultimate Estimate of the Upper Norm Bound for the Summation of Operators

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[1]Research supported by an NSERC grant.
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Abstract

Let A and B be bounded linear operators acting on a Hilbert space H. It is shown that the triangular inequality serves as the ultimate estimate of the upper norm bound for the sum of two operators in the sense that

$$\sup\{\|U^*AU + V^*BV\| : U \text{ and } V \text{ are unitaries}\} = \min\{\|A + \mu I\| + \|B - \mu I\| : \mu \in \mathbf{C}\}.$$

Consequences of the result related to spectral sets, the von Neumann inequality, and normal dilations are discussed. Furthermore, it is shown that the above equality can be used to characterize those unitarily invariant norms that are multiples of the operator norm in the finite dimensional case.

Keywords: Triangle inequalities, operator norm, unitarily invariant norm, normal dilations, spectral circles.

AMS Classifications: 47A30, 47A12, 15A60.

1 Introduction

Let H be a Hilbert space equipped with the inner product (x, y), and let $B(H)$ be the algebra of bounded linear operators acting on H equipped with the operator norm

$$\|A\| = \sup\{\|Ax\| : x \in H,\ (x, x) = 1\}.$$

If H is n-dimensional, we identify H with $\mathbf{C}^n$ and $B(H)$ with the algebra M_n of $n \times n$ complex matrices.

Basically, the *triangle inequality*

$$\|A + B\| \leq \|A\| + \|B\|$$

plays an important role in structure theory concerning the summation of matrices. In spite of the complexity of the norm computation, we will show that there are effective ways to obtain the best norm estimate for the sum of two operators.

For any $A, B \in B(H)$, it is clear that

$$\|U^*AU + V^*BV\| \leq \|U^*AU + \mu I\| + \|V^*BV - \mu I\| = \|A + \mu I\| + \|B - \mu I\|$$

for all $\mu \in \mathbf{C}$ and unitary $U, V \in B(H)$. We show that this rather trivial inequality is the ultimate estimate of the upper norm bound for $A + B$ in the sense that

$$\sup\{\|U^*AU + V^*BV\| : U \text{ and } V \text{ are unitaries}\} = \min\{\|A + \mu I\| + \|B - \mu I\| : \mu \in \mathbf{C}\}. \quad (1.1)$$

The inequality (1.1) is of great significance even when A and B are normal matrices. As established in [3], a sharp bound is obtained for $\|A_1 + iA_2\|$ where A_1 and A_2 are $n \times n$ Hermitian matrices satisfying $b_1 I \leq A_1 \leq c_1 I$ and $b_2 I \leq A_2 \leq c_2 I$.

Evidently, if the unitary similarity orbit of $A \in B(H)$ is the collection of operators unitarily similar to A, then the quantity in (1.1) can be viewed as a measure of (or a bound on) the distance between the unitary similarity orbits of A and $-B$. In particular, replacing B by $-B$ and μ by $-\mu$, we can rewrite equation (1.1) as

$$\sup\{\|U^*AU - V^*BV\| : U \text{ and } V \text{ are unitaries}\} = \min\{\|A + \mu I\| + \|B + \mu I\| : \mu \in \mathbf{C}\}.$$

Note that the supremum on the left side of (1.1) may not be attainable in the infinite-dimensional case; see [3, Example 5.1] for the full justification of the following example.

Example 1.1 Consider $A = \operatorname{diag}(0, 1/2, 2/3, 3/4, \ldots)$ and $B = \operatorname{diag}(1, 0, 0, \ldots)$ acting on $H = \ell_2$. Then

$$\begin{aligned} 2 &= \|A\| + \|B\| \\ &= \min\{\|A + \mu I\| + \|B - \mu I\| : \mu \in \mathbf{C}\} \\ &= \sup\{\|U^*AU + V^*BV\| : U \text{ and } V \text{ are unitaries}\}, \end{aligned}$$

and the supremum is not attainable.

We prove the equality (1.1) and show that the quantity in (1.1) is the same as

$$\sup\{\|AX + XB\| : X \in B(H),\ \|X\| \leq 1\}.$$

This may lead to a more direct proof of the result of Stampfli [7] concerning the norm of derivations. Furthermore, the quantity in (1.1) is also the same as

$$\sup\left\{\left\|\begin{pmatrix}(Ax,x)\\ \{\|Ax\|^2-|(Ax,x)|^2\}^{1/2}\end{pmatrix}+\begin{pmatrix}(By,y)\\ \{\|By\|^2-|(By,y)|^2\}^{1/2}\end{pmatrix}\right\|: x,y\in H,\ \|x\|=\|y\|=1\right\},$$

which is sort of optimal value of summation of two shells associated with A and B.

The equality in (1.1) may not hold if the Hilbert-space operator norm $\|\cdot\|$ is replaced by other norms.

Example 1.2 Consider the Frobenius norm ν on M_2, i.e., $\nu(T)=\{\mathrm{tr}\,(T^*T)\}^{1/2}$. Let

$$A=\begin{pmatrix}0&1\\1&0\end{pmatrix}\quad\text{and}\quad B=\begin{pmatrix}0&1\\-1&0\end{pmatrix}.$$

Then for any two unitaries $U,V\in M_2$, we have

$$\mathrm{tr}\,(U^*A^*UV^*BV+V^*B^*VU^*AU)=\mathrm{tr}\,(V^*(B+B^*)VU^*AU)=0,$$

and hence

$$\nu(U^*AU+V^*BV)=\{\mathrm{tr}\,(U^*A^*AU+V^*B^*BV)\}^{1/2}=2.$$

For any $\mu\in\mathbf{C}$, we have

$$\nu(A+\mu I)\geq\nu(A)=\sqrt{2}\quad\text{and}\quad\nu(B-\mu I)\geq\nu(B)=\sqrt{2}.$$

Thus

$$\begin{aligned}&\sup\{\nu(U^*AU+V^*BV): U \text{ and } V \text{ are unitaries}\}\\ =\ &2<2\sqrt{2}=\min\{\nu(A+\mu I)+\nu(B-\mu I):\mu\in\mathbf{C}\}.\end{aligned}$$

We will show that condition (1.1) can actually be used to characterize unitarily invariant norms on M_n that are multiples of the operator norm.

Suppose H' is a closed subspace of H, and P is the orthogonal projection of H onto H'. Then the operator $A'=PA|_{H'}:H'\to H'$ is a *compression* of A (actually, A' is *the compression* of A on H'), and A is called a *dilation* of A'.

Our paper is organized as follows. We prove our main theorem and some related results in Section 2. Some consequences of the main theorem related to spectral sets, the von Neumann inequality, and normal dilations are discussed in Section 3. In section 4, we use condition (1.1) to characterize the operator norm on M_n.

2 The main theorem

For an operator $T\in B(H)$, each unit vector $x\in H$ determines a vector $Tx=\alpha x+bx'$ with $\alpha\in\mathbf{C}$, $b\in[0,\infty)$, and x' as a unit vector orthogonal to x. Hence,

$$\begin{pmatrix}\alpha\\ b\end{pmatrix}=\begin{pmatrix}(Tx,x)\\ \{\|Tx\|^2-|(Tx,x)|^2\}^{1/2}\end{pmatrix}$$

is a vector in $\mathbf{C} \times \mathbf{R}$ of the same length as $\|Tx\|$ and, the vector $x' \in H$ is uniquely determined by x if $b \neq 0$ (equivalently, when x is not an eigenvector for T). Thus, the set

$$\Omega(T) = \left\{ \begin{pmatrix} (Tx, x) \\ \{\|Tx\|^2 - |(Tx,x)|^2\}^{1/2} \end{pmatrix} : x \in H,\ (x,x) = 1 \right\} \subseteq \mathbf{C} \times [0, \infty) \tag{2.1}$$

is a sort of *shell* associated with T capturing main effect of the norm and the quadratic form. (Cf. the notion of a *shell* as introduced by Davis [4].) For a further exploration on $\Omega(T)$, we note that

$$\Omega(T + \mu I) = \left\{ u + \begin{pmatrix} \mu \\ 0 \end{pmatrix} : u \in \Omega(T) \right\}$$

because

$$\{\|Tx\|^2 - |(Tx,x)|^2\}^{1/2} = \{\|(T+\mu I)x\|^2 - |((T+\mu I)x, x)|^2\}^{1/2}.$$

Thus,

$$\Omega(A) + \Omega(B) = \Omega(A + \mu I) + \Omega(B - \mu I)$$

and so

$$\sup\{\|u+v\| : u \in \Omega(A), v \in \Omega(B)\} = \sup\{\|u+v\| : u \in \Omega(A+\mu I), v \in \Omega(B - \mu I)\},$$

for all $\mu \in \mathbf{C}$.

The following is the statement of the main theorem.

Theorem 2.1 *Let $A, B \in B(H)$. Then*

$$\sup\{\|U^*AU + V^*BV\| : U \text{ and } V \text{ are unitaries}\} = \min\{\|A + \mu I\| + \|B - \mu I\| : \mu \in \mathbf{C}\}.$$

Moreover, the quantity in the above equality is the same as

$$\sup\{\|AX + XB\| : X \in B(H),\ \|X\| \le 1\},$$

which is also the same as

$$\sup\{\|u+v\| : u \in \Omega(A), v \in \Omega(B)\}.$$

Preparation for the proof of the main theorem

Let e_1, e_2 be two orthogonal vectors of length one in H. Suppose $u \in \Omega(A)$ and $v \in \Omega(B)$. Then there exist unitary $U, V \in B(H)$ such that

$$u = \begin{pmatrix} (U^*AUe_1, e_1) \\ (U^*AUe_1, e_2) \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} (V^*BVe_1, e_1) \\ (V^*BVe_1, e_2) \end{pmatrix}.$$

Thus

$$\|u + v\| = \|(U^*AU + V^*BV)e_1\| \le \|U^*AU + V^*BV\|,$$

and hence

$$\sup\{\|u+v\| : u \in \Omega(A),\ v \in \Omega(B)\} \le \sup\{\|U^*AU + V^*BV\| : U \text{ and } V \text{ are unitaries}\}.$$

Clearly, for any contraction $X \in B(H)$ and $\mu \in \mathbf{C}$,

$$\|AX + XB\| \le \|(A + \mu I)X\| + \|X(B - \mu I)\| \le \|A + \mu I\| + \|B - \mu I\|.$$

Hence,

$$\begin{aligned}
& \sup\{\|U^*AU + V^*BV\| : U \text{ and } V \text{ are unitaries}\} \\
= \; & \sup\{\|AUV^* + UV^*B\| : U \text{ and } V \text{ are unitaries}\} \\
\leq \; & \sup\{\|AX + XB\| : \|X\| \leq 1\} \\
\leq \; & \min\{\|A + \mu I\| + \|B - \mu I\| : \mu \in \mathbf{C}\}.
\end{aligned}$$

So, it remains to prove the following

Main Inequality

$$\min\{\|A + \mu I\| + \|B - \mu I\| : \mu \in \mathbf{C}\} \leq \sup\{\|u + v\| : u \in \Omega(A),\ v \in \Omega(B)\}. \tag{2.2}$$

We need some auxiliary results to prove this inequality. Denote by M_n^+ the set of positive semi-definite matrices. It is well known that M_n^+ is a convex cone, and the extreme rays are rank one matrices.

Lemma 2.2 *Let m be a positive integer smaller than 4. Suppose $\mathcal{S}$ is the intersection of M_n^+ and m real hyperplanes of the space of $n \times n$ Hermitian matrices; i.e., there are Hermitian matrices $F_1, \ldots, F_m$ and $\gamma_1, \ldots, \gamma_m \in \mathbf{R}$ such that*

$$\mathcal{S} = \{A \in M_n^+ : \operatorname{tr} AF_j = \gamma_j, j = 1, \ldots, m\}.$$

Then each extreme point of the convex set $\mathcal{S}$ has rank at most one.

Proof. Suppose $P \in \mathcal{S}$ has rank k such that $k > 1$. Let $P = RR^*$ such that R is $n \times k$. Then the real linear space

$$\mathbf{U} = \{RQR^* : Q^* = Q \in M_k\}$$

has real dimension $k^2 > 3 \geq m$. Thus the subspace

$$\mathbf{V} = \{X \in \mathbf{U} : \operatorname{tr} XF_j = 0, j = 1, \ldots, m\}$$

is nonzero. So, there is a nonzero $H \in \mathbf{V}$ such that both $P + H$ and $P - H$ are in M_n^+. It follows that P, as the average of $P + H$ and $P - H$ in $\mathcal{S}$ is not an extreme point. □

Lemma 2.3 *Suppose $A \in M_n$ and $\phi : M_n \to \mathbf{C}$ is a linear contractive map such that $\phi(A) = \|A\|$. Then there is a unit vector $x \in \mathbf{C}^n$ such that $\|Ax\| = \|A\|$ and $(Ax, x) = \overline{\phi(I)}\|A\|$.*

Proof. Without loss of generality, we may assume that A is a nonzero operator and $\|A\| = 1$ (otherwise replace A by $A/\|A\|$). By the Riesz representation, there is $C \in M_n$ such that $\phi(X) = \operatorname{tr}(CX)$ for all $X \in M_n$. Consider the polar decomposition $C = PU$ with $P \in M_n^+$ and U unitary. Then

$$\operatorname{tr} P = \phi(U^*) \leq \|U^*\| = 1 = \phi(A) = \operatorname{tr}(PUA).$$

Letting $X = U^*P^{1/2}$ and $Y = AP^{1/2}$, we have $\operatorname{tr}(X^*Y) = \operatorname{tr}(PUA) = 1$, $\operatorname{tr}(X^*X) = \operatorname{tr}(P) \leq 1$, $\operatorname{tr}(Y^*Y) = \operatorname{tr}(P^{1/2}A^*AP^{1/2}) \leq \operatorname{tr}(P) \leq 1$, and hence $\operatorname{tr}(X - Y)^*(X - Y) \leq 0$; thus $X = Y$, $\operatorname{tr}(P) = \operatorname{tr}(X^*X) = \operatorname{tr}(X^*Y) = 1$, and $\operatorname{tr}(AP) = \operatorname{tr}(U^*P) = \overline{\phi(I)}$.

From the fact $P^2 = P^{1/2}X^*XP^{1/2} = P^{1/2}Y^*YP^{1/2} = PA^*AP$, it follows that the range of P is a k-dimensional linear subspace of $\{v \in \mathbf{C}^n : \|Av\| = \|v\|\}$ with $1 \le k \le n$. Let R be an $n\times k$ matrix such that $R^*R = I_k$ and RR^* is the projection onto the range of P. Then $PRR^* = P = RR^*P$, and R^*PR is a matrix in

$$\mathcal{S} = \{Q \in M_k^+ : \operatorname{tr} Q = 1,\ \operatorname{tr}(R^*ARQ) = \overline{\phi(I)}\},$$

which is a non-empty compact convex set in the space of $k \times k$ Hermitian matrices obtained by intersecting M_k^+ with three real hyperplanes. By Lemma 2.2, $\mathcal{S}$ contains a rank-1 matrix yy^* with $y \in \mathbf{C}^k$ such that $\|y\|^2 = \operatorname{tr}(yy^*) = 1$ and $(R^*ARy, y) = \overline{\phi(I)}$. Letting $x = Ry$, we get all desired conditions of x. □

Proposition 2.4 *Let A and B be nonzero $n \times n$ matrices. The following are equivalent.*

(a) *There exist unit vectors $x, y \in \mathbf{C}^n$ such that*

$$\|Ax\| = \|A\|,\ \|By\| = \|B\|,\ \text{and } (Ax,x)/\|A\| = (By,y)/\|B\|.$$

(b) *There exist unit vectors $x, y \in \mathbf{C}^n$ such that $\|Ax\| = \|A\|$, $\|By\| = \|B\|$, and*

$$\|Ax\| + \|By\| \le \|(A+\mu I)x\| + \|(B-\mu I)y\| \quad \text{for all } \mu \in \mathbf{C}.$$

(c) $\|A\| + \|B\| \le \|A+\mu I\| + \|B - \mu I\|$ *for all $\mu \in \mathbf{C}$.*

Proof. (a) ⇒ (b): Suppose x and y satisfy condition (a). Then

$$\begin{aligned}
&\|Ax\| + \|By\| \\
= \;& \left\|\begin{pmatrix} (Ax,x) \\ \{\|Ax\|^2 - |(Ax,x)|^2\}^{1/2}\end{pmatrix}\right\| + \left\|\begin{pmatrix} (By,y) \\ \{\|By\|^2 - |(By,y)|^2\}^{1/2}\end{pmatrix}\right\| \\
= \;& \left\|\begin{pmatrix} (Ax,x) \\ \{\|Ax\|^2 - |(Ax,x)|^2\}^{1/2}\end{pmatrix} + \begin{pmatrix} (By,y) \\ \{\|By\|^2 - |(By,y)|^2\}^{1/2}\end{pmatrix}\right\| \\
= \;& \left\|\begin{pmatrix} ((A+\mu I)x,x) \\ \{\|(A+\mu I)x\|^2 - |((A+\mu I)x,x)|^2\}^{1/2}\end{pmatrix} + \begin{pmatrix} ((B-\mu I)y,y) \\ \{\|(B-\mu I)y\|^2 - |((B-\mu I)y,y)|^2\}^{1/2}\end{pmatrix}\right\| \\
\le \;& \left\|\begin{pmatrix} ((A+\mu I)x,x) \\ \{\|(A+\mu I)x\|^2 - |((A+\mu I)x,x)|^2\}^{1/2}\end{pmatrix}\right\| + \left\|\begin{pmatrix} ((B-\mu I)y,y) \\ \{\|(B-\mu I)y\|^2 - |((B-\mu I)y,y)|^2\}^{1/2}\end{pmatrix}\right\| \\
= \;& \|(A+\mu I)x\| + \|(B-\mu I)y\|.
\end{aligned}$$

(b) ⇒ (c): Clear.

(c) ⇒ (a): Consider the normed linear space $(M_n \times M_n, \nu)$ such that

$$\nu(X,Y) = \|X\| + \|Y\|.$$

Then the linear functional f on $\operatorname{span}\{(A,B),(I,-I)\}$ defined by $f(A,B) = \|A\| + \|B\|$ and $f(I,-I) = 0$ is contractive with respect to ν if and only if (c) holds. By the Hahn-Banach Theorem, f can be extended to a contractive linear functional F on $M_n \times M_n$. Since

$$\|A\| + \|B\| = F(A,B) \le |F(A,0)| + |F(0,B)| \le \|A\| + \|B\|,$$

it follows that

$$F(A,0) = \|A\| \quad \text{and} \quad F(0,B) = \|B\|.$$

Now, $X \mapsto F(X,0)$ is contractive. By Lemma 2.3, there is a unit vector $x \in \mathbf{C}^n$ such that

$$\|Ax\| = \|A\| \qquad \text{and} \qquad (Ax,x)/\|A\| = \overline{F(I,0)}.$$

Similarly, there exists a unit vector $y \in \mathbf{C}^n$ such that $\|By\| = \|B\|$ and $(By,y)/\|B\| = \overline{F(0,I)}$. From the fact

$$0 = F(I,-I) = F(I,0) - F(0,I),$$

we have $F(I,0) = F(0,I)$ and condition (a) holds. □

Proposition 2.4 actually holds for $A \in M_n$ and $B \in M_m$ with $n \neq m$. Of course, we then have $x \in \mathbf{C}^n$ and $y \in \mathbf{C}^m$ in conditions (a) and (b).

We are now ready to prove the main inequality (2.2)**.**

If A or B is a scalar operator, the result is clear. We assume that neither A nor B is scalar. First consider the finite dimensional case. Suppose

$$\|A + \mu_0 I\| + \|B - \mu_0 I\| \leq \|A + \mu I\| + \|B - \mu I\| \quad \text{for all } \mu \in \mathbf{C}.$$

In view of the fact $\Omega(A)+\Omega(B) = \Omega(A+\mu I)+\Omega(B-\mu I)$, we may assume that $\mu_0 = 0$ for simplicity. By Proposition 2.4, there exist unit vectors x and y in $\mathbf{C}^n$ such that

$$\|Ax\| = \|A\|, \quad \|By\| = \|B\|, \quad \text{and} \quad (Ax,x)/\|A\| = (By,y)/\|B\|.$$

Letting

$$u = \begin{pmatrix} (Ax,x) \\ \{\|Ax\|^2 - |(Ax,x)|^2\}^{1/2} \end{pmatrix} \in \Omega(A), \quad v = \begin{pmatrix} (By,y) \\ \{\|By\|^2 - |(By,y)|^2\}^{1/2} \end{pmatrix} \in \Omega(B)$$

we get

$$\|u + v\| = \|u\| + \|v\| = \|Ax\| + \|By\| = \|A\| + \|B\|$$

as desired.

Next, we consider the infinite dimensional case. Suppose the main inequality (2.2) is not true; i.e., there exists a positive real number ε such that

$$\sup\{\|u+v\| : u \in \Omega(A), v \in \Omega(B)\} < \|A + \mu I\| + \|B - \mu I\| - \varepsilon$$

for all $\mu \in \mathbf{C}$. We can find finitely many complex numbers $\lambda_1, \ldots, \lambda_m$ such that

$$\{\mu \in \mathbf{C} : |\mu| \leq \|A\| + \|B\|\} \subseteq \cup_{j=1}^m \{\lambda \in \mathbf{C} : |\lambda - \lambda_j| < \varepsilon/4\}.$$

Choose unit vectors $x_1, \ldots, x_m$ and $y_1, \ldots, y_m$ in H such that

$$\|(A + \lambda_j I)x_j\| > \|A + \lambda_j I\| - \varepsilon/4 \quad \text{and} \quad \|(B - \lambda_j I)y_j\| > \|B - \lambda_j I\| - \varepsilon/4$$

for each $j = 1, \ldots, m$. Let H' be the finite-dimensional subspace of H spanned by the $4m$ vectors $x_1, \ldots, x_m$, $y_1, \ldots, y_m$, and $Ax_1, \ldots, Ax_m$, $By_1, ..By_m$, and let A', B' and I' be the compressions of A, B and I on H'. Applying the finite dimensional result on (A', B'), we have

$$\begin{aligned} \min\{\|A' + \mu I'\| + \|B' - \mu I'\| : \mu \in \mathbf{C}\} &= \sup\{\|u+v\| : u \in \Omega(A'), v \in \Omega(B')\} \\ &\leq \sup\{\|u+v\| : u \in \Omega(A), v \in \Omega(B)\}. \end{aligned} \tag{2.3}$$

On the other hand, for each complex number μ with $|\mu| \le \|A\| + |B\|$, there exists j so that $|\mu - \lambda_j| < \varepsilon/4$ and thus

$$\|A' + \mu I'\| > \|A' + \lambda_j I'\| - \varepsilon/4 \ge \|(A' + \lambda_j I')x_j\| - \varepsilon/4$$

$$= \|(A + \lambda_j I)x_j\| - \varepsilon/4 > \|A + \lambda_j I\| - \varepsilon/2$$

and similarly,

$$\|B' - \mu I'\| > \|B - \lambda_j I\| - \varepsilon/2;$$

so

$$\|A' + \mu I'\| + \|B' - \mu I'\| > \sup\{\|u + v\| : u \in \Omega(A), v \in \Omega(B)\}.$$

Also for the case $|\mu| > \|A\| + \|B\|$, we have

$$\|A' + \mu I'\| + \|B' - \mu I'\| \ge \|2\mu I'\| - \|A'\| - \|B'\| > \|A\| + \|B\| > \sup\{\|u+v\| : u \in \Omega(A), v \in \Omega(B)\}.$$

Hence, there is a contradiction to (2.3); therefore, the main inequality (2.2) is true. □

3 Some consequences and related inequalities

3.1 Immediate corollaries

We can get many different formulas by putting special operators B in Theorem 2.1. For example, the substitution of (B, μ) by $(-B, -\mu)$ yields the following equalities:

$$\begin{aligned} & \sup\{\|U^*AU - V^*BV\| : U \text{ and } V \text{ are unitaries}\} \\ = \;& \min\{\|A - \mu I\| + \|B - \mu I\| : \mu \in \mathbf{C}\} \;=\; \sup\{\|AX - XB\| : X \in B(H),\ \|X\| \le 1\} \end{aligned}$$

for any $A, B \in B(H)$. The second quantity is a measure of distance to indicate how near is the pair (A, B) to the closest scalar operator, while the first quantity is a measure of the largest distance between two unitary similarity orbits.

Setting $B = -A$ with $-\mu$ in place of μ, we get the following equalities relating the diameter (maximum distance between all pairs of elements) of the unitary similarity orbit of A, the distance from A to the nearest scalar operator, and the operator norm of the derivation operator defined by $X \mapsto AX - XA$ (see [1]):

$$\begin{aligned} & \sup\{\|U^*AU - V^*AV\| : U \text{ and } V \text{ are unitaries}\} \\ = \;& 2\min\{\|A - \mu I\| : \mu \in \mathbf{C}\} \;=\; \sup\{\|AX - XA\| : X \in B(H),\ \|X\| \le 1\}. \end{aligned}$$

Furthermore, let $B = -e^{it}A$ for each $t \in [0, 2\pi)$, and define

$$\begin{aligned} g_A(t) &= \sup\{\|U^*AU - e^{it}V^*AV\| : U \text{ and } V \text{ are unitaries}\} \\ &= \min\{\|A - \mu I\| + \|e^{it}A - \mu I\| : \mu \in \mathbf{C}\} \\ &= \min\{\|A - \mu e^{it/2} I\| + \|A - \mu e^{-it/2} I\| : \mu \in \mathbf{C}\}. \end{aligned}$$

Then g_A is a continuous function satisfying $g_A(-t) = g_A(t)$ for all t, and

$$2\|A\| = g_A(\pi) \ge g_A(t) \ge g_A(0) = 2 \min_{\mu \in \mathbf{C}} \|A - \mu I\|.$$

Note that g_A is a monotone function on the interval $[0, \pi]$ in view of the triangle inequalities

$$\|T + \lambda I\| + \|T - \lambda I\| \geq r\|T + \lambda I\| + 2(1 - r)\|T\| + r\|T - \lambda I\| \geq \|T + r\lambda I\| + \|T - r\lambda I\|$$

for each real number $r \in [0, 1]$. Specifically, let θ be a fixed real number in $(0, \pi/2)$ and suppose

$$g_A(2\theta) = \|A - \alpha e^{i\theta} I\| + \|A - \alpha e^{-i\theta} I\|,$$

where α is a complex number. Letting

$$T = A - \alpha \cos(\theta) I, \qquad \text{and} \qquad \lambda = -i\alpha \sin(\theta),$$

we deuce that

$$g_A(2\theta) = \|T + \lambda I\| + \|T - \lambda I\| \geq \|T + r\lambda I\| + \|T - r\lambda I\| = \|A - \beta_+\| + \|A - \beta_-\|$$

with

$$\beta_\pm = \alpha(\cos(\theta) \pm ir \sin(\theta)),$$

and hence β_+/β_- is a complex number of modulus 1 with its argument ranging over the whole interval $[0, 2\theta]$ for $r \in [0, 1]$. Therefore g_A is a monotone function.

The following corollaries are statements about some delicate situations of the equality cases for some simple inequalities.

Corollary 3.1 *Let $A, B \in B(H)$. Then*

$$\min\{\|A + \mu I\| + \|B - \mu I\| : \mu \in \mathbf{C}\} \leq \|A\| + \|B\|.$$

The equality holds if and only if there exists a sequence of unitary operators $U_1, U_2, \ldots$ such that

$$\|A\| + \|B\| = \lim_{m \to \infty} \|A + U_m^* B U_m\|. \tag{3.1}$$

Corollary 3.2 *Let $A, B \in B(H)$. Then*

$$\|A + B\| \leq \sup\{\|U^*AU + V^*BV\| : U \text{ and } V \text{ are unitaries}\}. \tag{3.2}$$

The equality holds if and only if there exists $\mu_0 \in \mathbf{C}$ such that

$$\|A + B\| = \|A + \mu_0 I\| + \|B - \mu_0 I\|. \tag{3.3}$$

Several remarks concerning the above two corollaries are in order. In the finite dimensional case, we can replace the terms in the sequence of unitary operators in (3.1) by a constant unitary operator (matrix); also, the supremum of (3.2) can be replaced by maximum.

3.2 Optimal spectral circles, unitary similarity orbits, and normal dilations

Theorem 2.1 has interesting implications to spectral sets, unitary similarity orbits, and dilations of operators.

For $\mu \in \mathbf{C}$ and $r \geq 0$ let $\Gamma(\mu; r) = \{z \in \mathbf{C} : |z - \mu| = r\}$. If $\|A - \mu I\| \leq r$, then applying the von Neumann inequality (e.g., see [8]) to an affine transformation of the unit circle, we have

$$\|f(A)\| \leq \max\{|f(z)| : z \in \Gamma(\mu; r)\} \tag{3.4}$$

for any polynomial $f(z)$.

Note that for each operator $A \in B(H)$, there is a unique choice of $\mu_0 \in \mathbf{C}$ and $r_0 \geq 0$ so that

$$r_0 = \|A - \mu_0 I\| \leq \|A - \mu I\| \quad \text{for every } \mu \in \mathbf{C}. \tag{3.5}$$

To see this, assume that the above inequality is true for $\mu_0 = \mu_1$ and μ_{-1} with $\mu_1 \neq \mu_{-1}$. Then for $\tilde{\mu} = (\mu_1 + \mu_{-1})/2$, we have

$$\begin{aligned} 2\|A - \tilde{\mu} I\|^2 &\geq \|A - \mu_1 I\|^2 + \|A - \mu_{-1} I\|^2 \\ &\geq \|(A - \mu_1 I)^*(A - \mu_1 I) + (A - \mu_{-1} I)^*(A - \mu_{-1} I)\| \\ &= \|2(A - \tilde{\mu} I)^*(A - \tilde{\mu} I) + \frac{|\mu_1 - \mu_{-1}|^2}{2} I\| \\ &= 2\|A - \tilde{\mu} I\|^2 + |\mu_1 - \mu_{-1}|^2/2, \end{aligned}$$

which is a contradiction.

By the above discussion, there is a unique optimal (with smallest radius) spectral circle $\Gamma(\mu_0; r_0)$ satisfying (3.4), where μ_0 and r_0 are determined by (3.5). Furthermore, applying Theorem 2.1 to the pair $(A, -A)$, we have

$$2r_0 = 2\|A - \mu_0 I\| = \sup\{\|U^*AU - V^*AV\| : U \text{ and } V \text{ are unitaries}\},$$

where the quantity at the right end is just the diameter of the unitary similarity orbit of A. In particular, if A is a normal operator, then the optimal spectral circle $\Gamma(\mu_0; r_0)$ is the unique circle with minimum radius enclosing the spectrum of A, denoted by $\sigma(A)$; i.e.,

$$r_0 = \min_{\mu \in \mathbf{C}} \max\{|\alpha - \mu| : \alpha \in \sigma(A)\} = \max\{|\alpha - \mu_0| : \alpha \in \sigma(A)\}.$$

We can further extend the above discussion to two operators $A, B \in B(H)$ and obtain the following theorem concerning their joint spectral circles in connection with the distance between their unitary similarity orbits.

Theorem 3.3 *Let $A, B \in B(H)$, and let $\mu_0 \in \mathbf{C}$ be such that*

$$\|A - \mu_0 I\| + \|B - \mu_0 I\| \leq \|A - \mu I\| + \|B - \mu I\| \quad \text{for all } \mu \in \mathbf{C}.$$

Set $r_1 = \|A - \mu_0 I\|$ and $r_2 = \|B - \mu_0 I\|$. Then

$$\sup\{\|A - U^*BU\| : U \text{ unitary}\} = r_1 + r_2 \tag{3.6}$$

and

$$\|f(A) + U^*g(B)U\| \leq \max_{z \in \Gamma(\mu_0; r_1)} |f(z)| + \max_{z \in \Gamma(\mu_0; r_2)} |g(z)| \tag{3.7}$$

for each unitary U and each pair of polynomials $f(z)$ and $g(z)$.

Note that (3.6) can be viewed as the equality case of (3.7) for $f(z) = z - \mu_0$ and $g(z) = \mu_0 - z$.

Proof. Suppose $A, B \in B(H)$, and μ_0, r_1, r_2 satisfy the hypotheses. Applying Theorem 2.1 to the pair $(A, -B)$, we see see that

$$\sup\{\|A - U^*BU\| : U \text{ unitary}\} = \|A - \mu_0 I\| + \|B - \mu_0 I\| = r_1 + r_2$$

as asserted.

By the von Neumann inequality, we see that

$$\|f(A) + U^*g(B)U\| \leq \|f(A)\| + \|g(B)\| \leq \max_{z\in\Gamma(\mu_0;r_1)} |f(z)| + \max_{z\in\Gamma(\mu_0;r_2)} |g(z)|. \qquad \square$$

The next proposition gives a description for the set of complex numbers μ_0 in the statement of Theorem (3.3).

Proposition 3.4 *Let $A, B \in B(H)$, and let $S(A, B)$ be the set of complex numbers μ_0 satisfying*

$$\|A - \mu_0 I\| + \|B - \mu_0 I\| \leq \|A - \mu I\| + \|B - \mu I\| \quad \text{for all } \mu \in \mathbf{C}.$$

Then $S(A, B)$ is a either a singleton or a closed line segment.

Proof. Evidently, the set $S(A, B)$ is compact. Next, we show that $S(A, B)$ is convex. To see this, suppose $\mu_1, \mu_2 \in S(A, B)$. Let $\mu_0 = s\mu_1 + (1 - s)\mu_2$ with $s \in (0, 1)$. Then

$$\begin{aligned} & \|A - \mu_0 I\| + \|B - \mu_0 I\| \\ \leq\ & s\{\|A - \mu_1 I\| + \|B - \mu_1 I\|\} + (1 - s)\{\|A - \mu_2 I\| + \|B - \mu_2 I\|\} \\ \leq\ & \|A - \mu I\| + \|B - \mu I\| \end{aligned}$$

for all $\mu \in \mathbf{C}$. Hence, $\mu_0 \in S(A, B)$.

Now, we claim that $S(A, B)$ cannot include any disk

$$D(\mu_0; r) = \{\mu \in \mathbf{C} : |\mu - \mu_0| \leq r\}$$

with $r > 0$. If $D(\mu_0; r) \subseteq S(A, B)$, we may assume further that $\mu_0 = 0$ with (A, B) in place of $(A - \mu_0 I, B - \mu_0 I)$; thus,

$$\|A\| + \|B\| = \|A - \mu I\| + \|B - \mu I\| \qquad \text{for all } \mu \in D(0; r). \tag{3.8}$$

As $D(0; r) \setminus \{0\}$ is a connected set, and the function $f : D(0; r) \setminus \{0\} \to \mathbf{R}$ defined by $f(\mu) = \|A - \mu I\| - \|A + \mu I\|$ is continuous and $-f(\mu) = f(-\mu)$, it follows that there exists $\mu' \neq 0$ such that $f(\mu') = 0$; i.e. $\|A - \mu' I\| = \|A + \mu' I\|$. By (3.8), we also get that $\|B - \mu' I\| = \|B + \mu' I\|$. But the inequalities

$$\begin{aligned} 2\|A - \mu' I\|^2 &= \|A - \mu' I\|^2 + \|A + \mu' I\|^2 \\ &\geq \|(A - \mu' I)^*(A - \mu' I) + (A + \mu' I)^*(A + \mu' I)\| \\ &= 2\|A^*A + |\mu'|^2 I\| \\ &= 2\left\{\|A\|^2 + |\mu'|^2\right\} \\ &> 2\|A\|^2 \end{aligned}$$

leads to $\|A-\mu' I\| > \|A\|$, and similarly, $\|B-\mu' I\| > \|B\|$; so we obtain

$$\|A\| + \|B\| < \|A-\mu' I\| + \|B-\mu' I\|,$$

a contradiction to (3.8). Therefore, we see that $S(A,B)$ is a point or a closed line segment. □

Recall that every contraction in $B(H)$ has a unitary dilation. Applying affine transformations, we see that if $A \in B(H)$, $\mu \in \mathbf{C}$ and $r \geq 0$ satisfy $\|A-\mu I\| \leq r$, then A has a normal dilation $\tilde{A}$ such that $\sigma(\tilde{A}) \subseteq \Gamma(\mu; r)$. Suppose $\tilde{A}$ and $\tilde{B}$ are normal dilations of A and B. We have

$$\sup\{\|U^*AU - V^*BV\| : U, V \text{ unitary}\} \leq \sup\{\|\tilde{U}^*\tilde{A}\tilde{U} - \tilde{V}^*\tilde{B}\tilde{V}\| : \tilde{U}, \tilde{V} \text{ unitary}\}; \tag{3.9}$$

i.e., the distance between the unitary orbits of A and B is not larger than that of their normal dilations. Nevertheless, the following theorem shows that there always exist appropriate normal dilations whose unitary orbits are not farther apart.

Proposition 3.5 *Suppose $A, B \in B(H)$. Then*

$$\begin{aligned} & \sup\{\|U^*AU - V^*BV\| : U \textit{ and } V \textit{ are unitaries}\} \\ = \; & \min \sup\{\|\tilde{U}^*\tilde{A}\tilde{U} - \tilde{V}^*\tilde{B}\tilde{V}\| : \tilde{U} \textit{ and } \tilde{V} \textit{ are unitaries}\}, \end{aligned}$$

where min *is taken over all possible normal dilations $\tilde{A}$ and $\tilde{B}$ of A and B on a common Hilbert space. Specifically, let $\mu_0 \in \mathbf{C}$ be such that*

$$\|A-\mu_0 I\| + \|B-\mu_0 I\| \leq \|A-\mu I\| + \|B-\mu I\| \quad \textit{for every } \mu \in \mathbf{C},$$

$r_1 = \|A-\mu_0 I\|$, and $r_2 = \|B-\mu_0 I\|$. Then the set

$$S = \{(\tilde{A}, \tilde{B}) : \tilde{A} \textit{ and } \tilde{B} \textit{ are normal dilations of } A \textit{ and } B \textit{ on a common Hilbert space with } \sigma(\tilde{A}) \subseteq \Gamma(\mu_0; r_1) \textit{ and } \sigma(\tilde{B}) \subseteq \Gamma(\mu_0; r_2)\}$$

is non-empty, and every pair $(\tilde{A}, \tilde{B}) \in S$ satisfies

$$\begin{aligned} r_1 + r_2 &= \sup\{\|U^*AU - V^*BV\| : U \textit{ and } V \textit{ are unitaries}\} \\ &= \sup\{\|\tilde{U}^*\tilde{A}\tilde{U} - \tilde{V}^*\tilde{B}\tilde{V}\| : \tilde{U} \textit{ and } \tilde{V} \textit{ are unitaries}\}. \end{aligned}$$

Proof. Let A, B, μ_0, r_1, r_2 satisfy the hypotheses. Applying Theorem 2.1 to the pair $(A, -B)$, we have

$$r_1 + r_2 = \sup\{\|U^*AU - V^*BV\| : U \text{ and } V \text{ are unitaries}\}. \tag{3.10}$$

By the discussion before the theorem, the set S is non-empty. Suppose $(\tilde{A}, \tilde{B}) \in S$. Then

$$\|\tilde{U}^*\tilde{A}\tilde{U} - \tilde{V}^*\tilde{B}\tilde{V}\| \leq \|\tilde{A} - \mu_0 I\| + \|\tilde{B} - \mu_0 I\| \leq r_1 + r_2.$$

Combining with (3.9) and (3.10), we get the conclusion. □

3.3 Computation of the optimal values

In this subsection, we consider the problem of computing the four common quantities in Theorem 2.1. In the finite dimensional case, we can determine/approximate the quantity

$$\sup\{\|u+v\| : u \in \Omega(A),\ v \in \Omega(B)\}$$

by constructing the sets $\Omega(A)$ and $\Omega(B)$. For example, we can use standard algorithm (see [6, Chapter 1]) to construct the numerical range $W(T) = \{(Tx, x) : x \in \mathbf{C}^n,\ (x, x) = 1\}$ of $T \in M_n$; then compute $(\mu_j, c_j)^t \in \Omega(T)$ for some selected grid points $\mu_j \in W(T)$.

On the other hand, the computation of

$$\min\{\|A + \mu I\| + \|B - \mu I\| : \mu \in \mathbf{C}\} \tag{3.11}$$

can be carried out for μ varying over a (small) compact region in $\mathbf{C}$. As hidden in the proof of Theorem 2.1 in the infinite dimensional case,

$$\|A + \mu_0 I\| + \|B - \mu_0 I\| = \min\{\|A + \mu I\| + \|B - \mu I\| : \mu \in \mathbf{C}\}$$

occurs only for $|\mu| \le \|A\| + \|B\|$. Actually, there is a much smaller region as shown in the next proposition wherein we denote by $w(T) = \sup\{|z| : z \in W(T)\}$ the numerical radius of $T \in B(H)$.

Proposition 3.6 *Let $A, B \in B(H)$. Suppose $\mu_0 \in \mathbf{C}$ satisfies*

$$\|A + \mu_0 I\| + \|B - \mu_0 I\| = \min\{\|A + \mu I\| + \|B - \mu I\| : \mu \in \mathbf{C}\}.$$

Then

$$|\mu_0| \le \max\{w(A), w(B)\}.$$

Proof. Suppose λ is a complex number with $|\lambda| > \max\{w(A), w(B)\}$. Then, there is a real number $r \in [0, 1)$ such that

$$r|\lambda| = \max\{(w(A), w(B)\}.$$

Let T stand for A or B; then for each unit vector $v \in H$,

$$r|\lambda|^2 \ge |\lambda||(Tv, v)| = |(\bar{\lambda}Tv, v)| \ge \pm\mathrm{Re}(\bar{\lambda}Tv, v).$$

Thus

$$2r|\lambda|^2 I \ge \pm\bar{\lambda}T \pm \lambda T^*,$$

and

$$(T \pm \lambda I)^*(T \pm \lambda I) - (T \pm r\lambda I)^*(T \pm r\lambda I) = (1-r)(2r|\lambda|^2 I \pm \bar{\lambda}T \pm \lambda T^*) + (1-r)^2|\lambda|^2 I \ge (1-r)^2|\lambda|^2 I;$$

hence,

$$\|T \pm \lambda I\| > \|T \pm r\lambda I\|.$$

This shows, in particular,

$$\|A + \lambda I\| + \|B - \lambda I\| > \|A + r\lambda I\| + \|B - r\lambda I\| \ge \|A + \mu_0 I\| + \|B - \mu_0 I\|;$$

so $\lambda \neq \mu_0$. Since λ is an arbitrary complex number satisfying $|\lambda| > \max\{w(A), w(B)\}$, it follows that

$$|\mu_0| \leq \max\{w(A), w(B)\}$$

as desired. □

Given $T \in B(H)$, computing $\|T\|$ is easier than computing $w(T)$. So, we can use the larger region

$$R' = \{z \in \mathbf{C} : |z| \leq \max\{\|A\|, \|B\|\}$$

instead of $R = \{z \in \mathbf{C} : |z| \leq \max\{w(A), w(B)\}\}$ to solve the minimization problem (3.11).

Note that for normal operators $A, B \in B(H)$, the computation of the quantity in Theorem 2.1 reduces to a study of an optimization problem on $\mathbf{C}$, (see [3, Theorem 4.3]).

Corollary 3.7 *Suppose $A, B \in B(H)$ are normal with spectra $\sigma(A)$ and $\sigma(B)$. Then*

$$\begin{aligned} & \sup\{\|U^*AU + V^*BV\| : U \text{ and } V \text{ are unitaries}\} \\ = \; & \min_{\mu \in \mathbf{C}} \max\{|\alpha + \mu| + |\beta - \mu| : \alpha \in \sigma(A), \beta \in \sigma(B)\}. \end{aligned}$$

4 Characterizations of the operator norm

Recall that a norm ν on $B(H)$ is a *unitarily invariant norm* if $\nu(UXV) = \nu(X)$ for any $X \in B(H)$ and unitary $U, V \in B(H)$. Clearly, the operator norm on $B(H)$ is such a norm. We will show that the optimal situation of the triangle inequality of two matrices can be used to characterize unitarily invariant norms which are multiples of the operator norm on M_n.

For $X \in M_n$ the singular values $s_1(X) \geq \cdots \geq s_n(X)$ are the nonnegative square roots of the eigenvalues of X^*X. We begin with some auxiliary results, which are of independent interest.

Lemma 4.1 *Let $A, B \in M_n$. We have*

$$\sum_{j=1}^{k} s_j(A+B) \leq \sum_{j=1}^{k} \left(s_j(A) + s_j(B)\right), \qquad k = 1, \ldots, n.$$

For each fixed $k \in \{1, \ldots, n\}$, the inequality becomes equality if and only if there are unitary matrices $X, Y \in M_n$ such that

$$X^*AY = A_1 \oplus A_2 \qquad \text{and} \qquad X^*BY = B_1 \oplus B_2$$

such that $A_1, B_1 \in M_k$ are positive semi-definite with eigenvalues $s_1(A) \geq \cdots \geq s_k(A)$ and $s_1(B) \geq \cdots \geq s_k(B)$, respectively. (Here, A_2 and B_2 will be absent if $k = n$.)

Proof. The first statement is the well known Ky Fan inequality; the characterization of the equality case is Proposition 1.1 in [2]. □

Lemma 4.2 *Let k be a fixed integer in $\{1,\ldots,n\}$. Then $C \in M_n$ satisfy*

$$\sum_{j=1}^{k} s_j(I+C) = \sum_{j=1}^{k} (s_j(I) + s_j(C)) \tag{4.1}$$

if and only if $s_1(C),\ldots,s_k(C)$ are eigenvalues (counting multiplicities) of C.

Proof. Suppose (4.1) holds. By Lemma 4.1, there are unitary matrices $X, Y \in M_n$ such that

$$X^*Y = U_1 \oplus U_2 \qquad \text{and} \qquad X^*CY = C_1 \oplus C_2$$

so that $U_1, C_1 \in M_k$ are positive semi-definite with eigenvalues $s_1(I) = \cdots = s_k(I)$ and $s_1(C) \geq \cdots \geq s_k(C)$, respectively. Thus, $U_1 = I_k$ and the first k columns of X are the same as those of Y. Therefore, $Y^*X = I_k \oplus W$ where $W \in M_{n-k}$ is unitary, and

$$X^*CX = (X^*CY)(Y^*X) = C_1 \oplus C_2W;$$

so $s_1(C) \geq \cdots \geq s_k(C)$ are eigenvalues of C. The converse is clear. □

We denote the standard basis for M_n by $\{E_{11}, E_{12}, \ldots, E_{nn}\}$.

Lemma 4.3 *Let ν be a unitarily invariant norm on M_n. For any $A \in M_n$ we have*

$$\nu(s_1(A)E_{11}) \leq \nu(A) \leq \nu(s_1(A)I).$$

Proof. The result follows from the fact that two matrices $X, Y \in M_n$ satisfy $\nu(X) \leq \nu(Y)$ for all unitarily invariant norms ν on M_n if and only if $\sum_{j=1}^{k} s_j(X) \leq \sum_{j=1}^{k} s_j(Y)$ for all $k = 1,\ldots,n$. We give a short proof in the following.

Let $X_1 = \sum_{j=1}^{n} s_j(A)E_{jj}$ and $X_2 = s_1(A)E_{11} - \sum_{j=2}^{n} s_j(A)E_{jj}$. Then

$$s_1(A)E_{11} = (X_1 + X_2)/2 \quad \text{and} \quad \nu(s_1(A)E_{11}) \leq (\nu(X_1) + \nu(X_2))/2 = \nu(A).$$

Also, let $A_0 = \left(\sum_{j=1}^{n} s_j(A)E_{jj}\right)/s_1(A)$. Then $Y_1 = A_0 + i\sqrt{I - A_0^2}$ and $Y_2 = A_0 - i\sqrt{I - A_0^2}$ are unitary matrices. Moreover, $A_0 = (Y_1 + Y_2)/2$ and

$$\nu(A)/s_1(A) = \nu(A_0) \leq (\nu(Y_1) + \nu(Y_2))/2 = \nu(I).$$

□

We need one more proposition to prove our main theorem.

Proposition 4.4 *Let ν be a unitarily invariant norm on M_n. The following are equivalent.*

(a) *ν is a (positive) multiple of the Hilbert-space operator norm.*

(b) *For every matrix $C \in M_n$ such that $s_1(C)$ is an eigenvalue of C, we have*

$$\nu(I + C) = \nu(I) + \nu(C).$$

(c) *There exists a rank-two matrix $C \in M_n$ which is not positive semi-definite and satisfies*

$$\nu(I + C) = \nu(I) + \nu(C).$$

(d) $\nu(I) = \nu(E_{11})$.

Note that in view of condition (b), we can choose special matrices C such as $C = E_{11} - E_{22}$ or $C = E_{11} + E_{23}$ to test the equality $\nu(I + C) = \nu(I) + \nu(C)$ in condition (c). Note also that if we put $C = \operatorname{diag}(1, -1, \ldots, -1)$ in condition (b), the implication "(b) $\Rightarrow$ (d)" follows immediately.

Proof of Proposition 4.4. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are clear. Suppose (c) holds; i.e., C is an $n \times n$ matrix satisfying

$$\nu(I + C) = \nu(I) + \nu(C)$$

and C has n singular values

$$c_1 \geq c_2 > c_3 = \cdots = c_n = 0,$$

where either c_1 or c_2 is not an eigenvalue of C. By Lemma 4.2, we have

$$s_1(I + C) \leq s_1(I) + s_1(C) = 1 + c_1,$$

$$\sum_{j=1}^{k} s_j(I + C) < \sum_{j=1}^{k} (s_j(I) + s_j(C)) = k + c_1 + c_2 \quad \text{for } k = 2, \ldots, n.$$

Thus, there exist a positive real number $\varepsilon < 1$ such that

$$\sum_{j=1}^{k} s_j(I + C) < 1 + (1 - \varepsilon)(k - 1) + c_1 + c_2 \quad \text{for } k = 2, \ldots, n.$$

Let D_0, D_1 and D_2 be three diagonal $n \times n$ matrices specified as

$$D_0 = c_1 E_{11} + c_2 E_{22}, \qquad D_1 = \sum_{j=1}^{n} s_j(I + C) E_{jj}, \qquad D_2 = (1 - \varepsilon) I + \varepsilon E_{11} + D_0.$$

Then $\nu(D_0) = \nu(C)$, $s_j(D_1) = s_j(I + C)$ while $s_1(D_2) = 1 + c_1$, $s_2(D_2) = (1 - \varepsilon + c_2)$, and $s_k(D_2) = 1 - \varepsilon$ for $k > 2$. By the result in [5], we get $\nu(D_1) \leq \nu(D_2)$, but

$$\nu(D_1) = \nu(I + C) = \nu(I) + \nu(C), \quad \nu(D_2) \leq (1 - \varepsilon)\nu(I) + \varepsilon\nu(E_{11}) + \nu(C),$$

and hence $\nu(I) \leq \nu(E_{11})$. As $2E_{11} - I$ is unitary, we also have

$$\nu(E_{11}) \leq \nu(I)/2 + \nu(2E_{11} - I)/2 = \nu(I),$$

and therefore $\nu(E_{11}) = \nu(I)$. Thus, condition (d) is established. □

We are now ready to present the main theorem of this section.

Theorem 4.5 *Let ν be a unitarily invariant norm on M_n. The following are equivalent.*

(a) *ν is a (positive) multiple of the Hilbert-space operator norm.*

(b) *For all pairs of matrices $A, B \in M_n$,*

$$\max\{\nu(A + U^* B U) : U \text{ unitary}\} = \min\{\nu(A + \mu I) + \nu(B - \mu I) : \mu \in \mathbf{C}\}. \qquad (4.2)$$

(c) *Equation* (4.2) *holds for the specified pair*

$$A = E_{12} + E_{23} + \cdots + E_{n-1,n} + E_{n,1} \qquad \text{and} \qquad B = \xi E_{11} - \xi E_{22},$$

where $\xi \in \mathbf{C}$ *with* $|\xi| = 1$ *and* $\xi^{2n} \neq 1$.

Proof. By Theorem 2.1, (a) ⇒ (b). The implication (b) ⇒ (c) is clear.

Suppose (c) holds. Let $\omega = e^{2\pi i/n}$. Then for $k = 1, \ldots, n-1$, there is a unitary matrix $V_k \in M_n$ such that $V_k^* A V_k = \omega^k A$. Thus for any $\mu \in \mathbf{C}$, we have

$$\nu(A + \mu I) = \nu(V_k^*(A + \mu I)V_k) = \nu(\omega^k A + \mu I) = \nu(A + \omega^{-k}\mu I),$$

and

$$\nu(A) \leq \frac{1}{n}\sum_{k=0}^{n-1} \nu(A + \omega^{-k}\mu I) = \nu(A + \mu I).$$

So,

$$\nu(A) = \min\{\nu(A + \mu I) : \mu \in \mathbf{C}\}.$$

Similarly, we can show that $\nu(B + \mu I) = \nu(B - \mu I)$ for all $\mu \in \mathbf{C}$ and hence

$$\nu(B) = \min\{\nu(B - \mu I) : \mu \in \mathbf{C}\}.$$

Thus,

$$\nu(A) + \nu(B) = \min\{\nu(A + \mu I) + \nu(B - \mu I) : \mu \in \mathbf{C}\}.$$

Assume that $U \in M_n$ is unitary satisfying the condition (c) so that

$$\nu(I) + \nu(A^*U^*BU) = \nu(A) + \nu(B) = \nu(A + U^*BU) = \nu(I + A^*U^*BU).$$

Note that $C = A^*U^*BU$ is a rank-2 matrix. We claim that $C \neq C^*$. If it is not true, then for $V = U^*BU$ we have

$$A^*V = C = C^* = V^*A = \bar{\xi}^2 VA.$$

Since $A^n = I_n$, we see that

$$V = A^{*n}V = (A^*)^{n-1}V(\bar{\xi}^2 A) = \cdots = V(\bar{\xi}^2 A)^n = \bar{\xi}^{2n}V.$$

Hence, $\bar{\xi}^{2n} = 1$, which is a contradiction. So, our claim is valid, and condition (c) of Proposition 4.4 holds. Therefore, ν is a multiple of the Hilbert-space operator norm, i.e., condition (a) holds. □

It would be nice to extend Theorem 4.5 to other class of norms or show that such an extension is impossible.

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