Linear Maps Leaving the Alternating Group Invariant

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Abstract

Let $\mathbf{A}_n$ be the group of $n \times n$ even permutation matrices, and let $\mathbf{V}_n$ be the real linear space spanned by $\mathbf{A}_n$. The purpose of this note is to characterize those linear operators ϕ on $\mathbf{V}_n$ satisfying $\phi(\mathbf{A}_n) = \mathbf{A}_n$. This answers a question raised by Li, Tam and Tsing.

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1 Introduction

Let $\mathbf{S}_n$ (respectively $\mathbf{A}_n$) be the group of $n \times n$ (respectively, even) permutation matrices, and let $\mathbf{V}_n$ be the real linear space spanned by $\mathbf{A}_n$. The purpose of this note is to characterize those linear operators ϕ on $\mathbf{V}_n$ satisfying $\phi(\mathbf{A}_n) = \mathbf{A}_n$. This answers an open problem in [3].

The problem for $\mathbf{A}_2$ is trivial since $\mathbf{A}_2$ is a singleton, and $\mathbf{V}_2$ is a one dimensional space. The set $\mathbf{A}_3$ consists of 3 linearly independent matrices, and thus $\mathbf{V}_3$ is 3-dimensional and $\mathbf{A}_3$ is a basis. A linear map ϕ on $\mathbf{V}_3$ satisfies $\phi(\mathbf{A}_3) = \mathbf{A}_3$ if and only if it permutes the elements in the basis $\mathbf{A}_3$.

For $n \geq 4$, we have the following results.

Theorem 1.1 *Suppose $n \geq 4$. Then $\mathbf{V}_n$ is the space of $n \times n$ real matrices with equal row sums and columns sums.*

Let $\mathbf{U}_n$ be the space of $n \times n$ real matrices with equal row sums and columns sums. Then $\operatorname{span} \mathbf{A}_n \subseteq \operatorname{span} \mathbf{S}_n \subseteq \mathbf{U}_n$. By Theorem 1.1, we have $\mathbf{U}_n = \operatorname{span} \mathbf{A}_n = \operatorname{span} \mathbf{S}_n$ if $n \geq 4$, and it is easy to see that $\operatorname{span} \mathbf{A}_n \neq \operatorname{span} \mathbf{S}_n = \mathbf{U}_n$ for $n \in \{2, 3\}$. In [3, Section 2] the authors used Birkhoff Theorem to deduce that $\operatorname{span} \mathbf{S}_n = \mathbf{U}_n$ for any positive integer n.

Theorem 1.2 *Consider the normal subgroup $\mathbf{H}_0 = \{P \in \mathbf{A}_4 : P^2 = I_4\}$ of $\mathbf{A}_4$, and the two cosets $\mathbf{H}_1$ and $\mathbf{H}_2$ of $\mathbf{H}_0$ in $\mathbf{A}_4$. If $\phi : \mathbf{V}_4 \to \mathbf{V}_4$ is a linear map such that $\phi(\mathbf{A}_4) = \mathbf{A}_4$, then*

$$\phi(\mathbf{H}_j) = \mathbf{H}_{i_j} \quad \text{for} \quad j = 0, 1, 2, \quad \text{with} \quad \{i_0, i_1, i_2\} = \{0, 1, 2\}. \tag{1}$$

Conversely, if ψ is a permutation on $\mathbf{A}_4$ such that (1) holds for $\phi = \psi$, then ψ can be extended uniquely to a linear map on $\mathbf{V}_4$.

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Theorem 1.3 *Let $n \geq 5$. A linear map $\phi : \mathbf{V}_n \to \mathbf{V}_n$ satisfies $\phi(\mathbf{A}_n) = \mathbf{A}_n$ if and only if there exist $P, Q \in \mathbf{S}_n$ with $PQ \in \mathbf{A}_n$ such that ϕ is of the form*

$$A \mapsto PAQ \qquad \text{or} \qquad A \mapsto PA^tQ. \tag{2}$$

It was proved in [3, Theorem 2.2] that linear operators on span $\mathbf{S}_n$ mapping $\mathbf{S}_n$ onto itself have the form (2) for some $P, Q \in \mathbf{S}_n$. If one can show that for $n \geq 5$ every linear operator ϕ on $\mathbf{V}_n$ that satisfies $\phi(\mathbf{A}_n) = \mathbf{A}_n$ also satisfies $\phi(\mathbf{S}_n) = \mathbf{S}_n$, then Theorem 1.3 will follow. However, there does not seem to be an easy proof of this.

For any linear operator ϕ on $\mathbf{V}_n$ satisfying $\phi(\mathbf{A}_n) = \mathbf{A}_n$, we can replace it by the linear operator ψ of the form $A \mapsto \phi(I_n)^{-1}\phi(A)$. Then ψ is *unital*, i.e., $\psi(I_n) = I_n$, and satisfies $\psi(\mathbf{A}_n) = \mathbf{A}_n$. Using this observation, one easily sees that Theorem 1.3 is equivalent to the following.

Theorem 1.4 *Let $n \geq 5$. A linear map $\phi : \mathbf{V}_n \to \mathbf{V}_n$ satisfies $\phi(I_n) = I_n$ and $\phi(\mathbf{A}_n) = \mathbf{A}_n$ if and only if there exists $P \in \mathbf{S}_n$ such that ϕ is of the form*

$$A \mapsto PAP^t \qquad \text{or} \qquad A \mapsto PA^tP^t,$$

i.e., the restriction of ϕ on $\mathbf{A}_n$ is a group automorphism or anti-automorphism.

2 Auxiliary Results and Proofs

Let $\{e_1, \ldots, e_n\}$ and $\{E_{11}, E_{12}, \ldots, E_{nn}\}$ be the standard bases for $\mathbb{R}^n$ and the linear space of $n \times n$ real matrices, respectively. We use the usual cycle notation to represent a permutation matrix in $\mathbf{S}_n$. For example, $(i, j) \in \mathbf{S}_n$ will represent the permutation matrix P obtained from I_n by interchanging the ith and jth rows. Every element $P \in \mathbf{S}_n$ can be regarded as a bijection $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$, and vice versa; namely, the matrix $P = [e_{\sigma(1)}| \cdots |e_{\sigma(n)}] \in \mathbf{S}_n$ corresponds to the bijection σ. We will use both interpretations in our discussion. For instance, if $P \in \mathbf{S}_n$ corresponds to a permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$, and $Q \in \mathbf{S}_n$ corresponds to the k-cycle $(i_1, \ldots, i_k)$, then PQP^t corresponds to the k-cycle $(\sigma(i_1), \ldots, \sigma(i_k))$.

Denote by J_n the $n \times n$ matrix with all entries equal to $1/n$. For $1 \leq k \leq n$, let

$$\mathbf{U}_k = \{X_0 \oplus \gamma I_{n-k} : \gamma \in \mathbb{R},\ X_0 \text{ is } k \times k \text{ with all row and column sums equal to } \gamma\}. \tag{3}$$

Then $\mathbf{U}_n$ is just the set of $n \times n$ real matrices with equal row sums and column sums defined in the last section, and Theorem 1.1 asserts that $\mathbf{U}_n = \mathbf{V}_n$. Indeed, every $n \times n$ matrix A with equal row sums and column sums γ can be written as $A = A_0 + \gamma I$ so that $A_0 = A - \gamma I$ has row sums and column sums zero. We have the following result, from which Theorem 1.1 readily follows.

Proposition 2.1 *Let $n \geq k \geq 4$. Then $\mathbf{U}_k$ defined as in (3) is spanned by elements in $\mathbf{A}_n$ of the form:*

$$R = (i_1, i_2)(i_3, i_4) \qquad \text{with} \qquad i_1, i_2, i_3, i_4 \in \{1, \ldots, k\}. \tag{4}$$

(Note that R can only be I_n, a 3-cycle, or a product of two disjoint transpositions.)

Proof. Since $\mathbf{U}_k$ has a basis

$$\mathcal{B} = \{I_n\} \cup \{E_{ij} - E_{ik} - E_{kj} + E_{kk} : 1 \le i, j \le k-1\},$$

it has dimension $(k-1)^2 + 1$.

When $n \ge k = 4$, $\mathbf{U}_4$ has dimension 10, and every matrix in $\mathbf{A}_n \cap \mathbf{U}_4$ is of the form (4) with $i_1, i_2, i_3, i_4 \in \{1,2,3,4\}$. One can check that there are 10 linearly independent matrices in $\mathbf{A}_n \cap \mathbf{U}_4$. This can also be done by a simple Matlab program as shown in the Appendix.

Suppose $n \ge k \ge 5$. It suffices to show that every element in $\mathcal{B}$ is a linear combination of matrices in $\mathbf{A}_n$ of the form (4). Clearly, I_n is of the form (4). For any $1 \le i, j \le k-1$, let $F_{ij} = E_{ij} - E_{ik} - E_{kj} + E_{kk}$, and let p, q, r, s be distinct elements of $\{1, \dots, k\}$ that satisfy $\{i, j, k\} \subseteq \{p, q, r, s\}$. Suppose Q is a permutation mapping the indices p, q, r, s to $1, 2, 3, 4$. Then $QF_{ij}Q^t \in \mathbf{U}_4$, and we can apply the result on $\mathbf{U}_4$ to conclude that $QF_{ij}Q^t$ is a linear combination of matrices $R_1, \dots, R_m$ of the form (4) with $i_1, i_2, i_3, i_4 \in \{1,2,3,4\}$. Thus, F_{ij} is a linear combination of the matrices $Q^tR_1Q, \dots, Q^tR_mQ$, all of the form (4) with $i_1, i_2, i_3, i_4 \in \{p, q, r, s\}$. □

We need the following lemma to prove Theorem 1.2.

Lemma 2.2 *Let $n \ge 3$. Suppose ϕ is a linear map on $\mathbf{V}_n$ satisfying $\phi(\mathbf{A}_n) = \mathbf{A}_n$. Then*

$$\phi(J_n) = J_n. \tag{5}$$

Proof. Since $\sum_{P \in \mathbf{A}_n} P = n!J_n/2$, we have

$$\phi(n!J_n) = \phi(2 \sum_{P \in \mathbf{A}_n} P) = 2 \sum_{P \in \mathbf{A}_n} \phi(P) = 2 \sum_{P \in \mathbf{A}_n} P = n!J_n.$$

The result follows. □

Proof of Theorem 1.2. Let $\mathbf{H}_0, \mathbf{H}_1, \mathbf{H}_2$ be defined as in the theorem. Clearly, $\mathbf{H}_0 = \{I_4, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$. We claim that:

$$\mathbf{T} = \{X_1, \dots, X_4\} \subseteq \mathbf{A}_4$$

satisfies $X_1 + \cdots + X_4 = 4J_4$ if and only if $\mathbf{T} = \mathbf{H}_i$ for some $i \in \{0,1,2\}$.

To prove this claim, let $\mathbf{T}$ be such a set. Then the matrices $Y_j = X_1^tX_j$ for $j \in \{1, \dots, 4\}$ satisfy $Y_1 = I_4$ and $Y_1 + \cdots + Y_4 = 4J_4$; so, Y_2, Y_3, Y_4 all have zero diagonals, and thus $\{Y_1, \dots, Y_4\} = \mathbf{H}_0$. Hence, $\mathbf{T} = X_1\mathbf{H}_0$ is a coset of $\mathbf{H}_0$ as asserted.

Suppose $\phi : \mathbf{V}_4 \to \mathbf{V}_4$ satisfies $\phi(\mathbf{A}_4) = \phi(\mathbf{A}_4)$. Let $\mathbf{H}_0 = \{X_1, \dots, X_4\}$. By Lemma 2.2, we have

$$4J_4 = \phi(4J_4) = \phi(X_1 + \cdots + X_4) = \phi(X_1) + \cdots + \phi(X_4).$$

It follows from our claim that $\phi(\mathbf{H}_0) = \mathbf{H}_{i_0}$ for some $i_0 \in \{0,1,2\}$. Repeating the arguments to $\mathbf{H}_1$ and $\mathbf{H}_2$, we see that $\phi(\mathbf{H}_j) = \mathbf{H}_{i_j}$ for $j = 0,1,2$, where $\{i_0, i_1, i_2\} = \{0,1,2\}$.

Suppose ψ permutes the elements in $\mathbf{A}_n$ and satisfies $\psi(\mathbf{H}_j) = \mathbf{H}_{i_j}$ for $j = 0, 1, 2$, where $\{i_0, i_1, i_2\} = \{0, 1, 2\}$. Take 3 elements from each of the set $\mathbf{H}_j$ for $j = 0, 1, 2$, to get 9 matrices $Y_1, \dots, Y_9 \in \mathbf{A}_4$. One can check, say, using Matlab, that $\{J_4, Y_1, \dots, Y_9\}$ is a basis for $\mathbf{V}_4$. Define the linear map $\phi : \mathbf{V}_4 \to \mathbf{V}_4$ by $\phi(J_4) = J_4$ and $\phi(Y_j) = \psi(Y_j)$ for $j = 1, \dots, 9$. Then ϕ is the unique linear operator on $\mathbf{V}_4$ such that $\phi(X) = \psi(X)$ for all $X \in \mathbf{A}_4$. $\square$

The rest of this section is devoted to proving Theorem 1.4. We need some more notations and lemmas. Let $\tilde{\mathbf{U}}_n$ be the set of $n \times n$ real matrices with row sums and column sums zero, and let

$$\tilde{\mathbf{A}}_n = \mathbf{A}_n - J_n = \{P - J_n : P \in \mathbf{A}_n\}.$$

Then $\tilde{\mathbf{A}}_n \subseteq \tilde{\mathbf{U}}_n$ is a group with $I_n - J_n$ as the identity, and $P^t - J_n$ as the inverse of $P - J_n$, for any $P \in \mathbf{A}_n$. Moreover, since $\tilde{\mathbf{U}}_n \subseteq \mathbf{U}_n = \mathbf{V}_n$, for any $X \in \tilde{\mathbf{U}}_n$ there is a linear combination of $P_1, \dots, P_k \in \mathbf{A}_n$ such that $\sum_{j=1}^k \alpha_j P_j = X$. Since X has zero row sums and column sums, we see that $\sum_{j=1}^k \alpha_j = 0$, and thus $X = \sum_{j=1}^k \alpha_j (P_j - J_n)$. Hence we have $\tilde{\mathbf{U}}_n = \operatorname{span} \tilde{\mathbf{A}}_n$. Suppose $\phi : \mathbf{V}_n \to \mathbf{V}_n$ is a linear map that satisfies $\phi(\mathbf{A}_n) = \mathbf{A}_n$. Since $\phi(J_n) = J_n$ by Lemma 2.2, and $\phi(P - J_n) = \phi(P) - J_n$ for any $P \in \mathbf{A}_n$, we have $\phi(\tilde{\mathbf{A}}_n) \subseteq \tilde{\mathbf{A}}_n$, and hence $\phi(\tilde{\mathbf{U}}_n) \subseteq \tilde{\mathbf{U}}_n$.

The usual inner product

$$(X, Y) = \operatorname{tr}(XY^t)$$

on $n \times n$ real matrices induces inner products on the subspaces $\mathbf{V}_n$ and $\tilde{\mathbf{U}}_n$. Let G be the group of linear operators ψ on $\tilde{\mathbf{U}}_n$ satisfying $\psi(\tilde{\mathbf{A}}_n) = \tilde{\mathbf{A}}_n$. Since $\tilde{\mathbf{A}}_n$ is a compact set spanning $\tilde{\mathbf{U}}_n$, we see that G is a compact group of nonsingular linear operators on $\tilde{\mathbf{U}}_n$. By a result of Auerbach [1] (see [2] for an elementary proof), there exists a positive definite operator T on $\tilde{\mathbf{U}}_n$ such that

$$TGT^{-1} = \{T\psi T^{-1} : \psi \in G\}$$

is a subgroup of $O(\tilde{\mathbf{V}}_n)$ – the group of orthogonal operators on $\tilde{\mathbf{U}}_n$. Denote by L^* the adjoint operator of L on $\tilde{\mathbf{U}}_n$, i.e., $(L(X), Y) = (X, L^*(Y))$ for all $X, Y \in \tilde{\mathbf{U}}_n$. Then for any $\psi \in G$, $(T\psi T^{-1})^*(T\psi T^{-1})$ is the identity operator on $\tilde{\mathbf{U}}_n$, i.e., $T^2\psi = (\psi^*)^{-1}T^2$.

Note that $\tilde{\mathbf{U}}_n$ and M_{n-1} are isomorphic algebras. To see this, consider an orthogonal matrix P whose last row equals $(1, \dots, 1)/\sqrt{n}$. Then for every $X \in \tilde{\mathbf{U}}_n$, we have $PXP^t = \hat{X} \oplus [0]$ with $\hat{X} \in M_{n-1}$, and the mapping $X \mapsto \hat{X}$ is an algebra isomorphism. It is well known (and easy to check) that if $\mathcal{S}$ is a spanning set for the linear space M_{n-1}, then the mappings of the form $X \mapsto PXQ$ with $P, Q \in \mathcal{S}$ span the linear space of linear transformations from M_{n-1} to itself. An analogous result holds for $\tilde{\mathbf{U}}_n$. So, if H is the subgroup of G consisting of operators of the form $X \mapsto PXQ$ with $P, Q \in \tilde{\mathbf{A}}_n$, then H spans the space of all linear operators on $\tilde{\mathbf{U}}_n$ because $\operatorname{span} \tilde{\mathbf{A}}_n = \tilde{\mathbf{U}}_n$. Furthermore, every element in H satisfies $\psi^* = \psi^{-1}$, and hence $T^2\psi = \psi T^2$ for all $\psi \in H$. It follows that T^2 commutes with all operators on $\tilde{\mathbf{U}}_n$; hence T^2 is a scalar operator. Since T is a positive definite operator, T is a scalar operator as well. Thus, we have $TGT^{-1} = G$ and so G is a subgroup of $O(\tilde{\mathbf{U}}_n)$, i.e., every element in G preserves the inner product on $\tilde{\mathbf{U}}_n$.

Consider any linear map $\phi : \mathbf{V}_n \to \mathbf{V}_n$ that satisfies $\phi(\mathbf{A}_n) = \mathbf{A}_n$. Suppose $\tilde{\phi}$ is the restriction of ϕ on $\tilde{\mathbf{U}}_n$ (to $\tilde{\mathbf{U}}_n$). For any $R, S \in \mathbf{V}_n$ with row sums r and s, respectively, we can write $R = R_0 + rJ_n$ and $S = S_0 + sJ_n$ with $R_0, S_0 \in \tilde{\mathbf{U}}_n$. By what we have just shown, $\tilde{\phi}$ preserves the inner product on $\tilde{\mathbf{U}}_n$. And since $\phi(J_n) = J_n$, we have

$$\begin{aligned}
(R, S) &= (R_0 + rJ_n, S_0 + sJ_n) \\
&= (R_0, S_0) + rs \\
&= (\tilde{\phi}(R_0), \tilde{\phi}(S_0)) + rs \\
&= (\tilde{\phi}(R_0) + rJ_n, \tilde{\phi}(S_0) + sJ_n) \\
&= (\phi(R_0) + rJ_n), \phi(S_0) + sJ_n) \\
&= (\phi(R_0 + rJ_n), \phi(S_0 + sJ_n)) \\
&= (\phi(R), \phi(S)).
\end{aligned}$$

Summarizing, we have the following lemma.

Lemma 2.3 *Suppose $n \geq 4$, and ϕ is a linear operator on $\mathbf{V}_n$ satisfying $\phi(\mathbf{A}_n) = \mathbf{A}_n$. Then*

$$(\phi(R), \phi(S)) = (R, S) \qquad \textit{for any } R, S \in \mathbf{V}_n. \tag{6}$$

Lemma 2.4 *Suppose $n \geq 4$.*

(a) *For any $S \in \mathbf{A}_n$, (I_n, S) is just the number of nonzero diagonal entries of S.*

(b) *For any $R, S \in \mathbf{A}_n$, $(R, S) = (I_n, R^t S)$.*

(c) *For any two different $R, S \in \mathbf{A}_n$, we have $(R, S) \leq n - 3$, where the equality holds if and only if $R^t S$ is a 3-cycle. Moreover,*

(c.1) *a 3-cycle (i_1, i_2, i_3) and a 5-cycle $(j_1, \ldots, j_5)$ have inner product $n - 3$ if and only if (i_1, i_2, i_3) is one of the following:*

$$(j_1, j_2, j_3), \quad (j_2, j_3, j_4), \quad (j_3, j_4, j_5), \quad (j_4, j_5, j_1), \quad (j_5, j_1, j_2);$$

(c.2) *a 5-cycle $(j_1, \ldots, j_5)$ and a product of two disjoint transpositions $(i_1, i_2)(i_3, i_4)$ have inner product $n - 3$ if and only if $(i_1, i_2)(i_3, i_4)$ is one of the following:*

$$(j_1, j_2)(j_3, j_4), \quad (j_1, j_2)(j_4, j_5), \quad (j_2, j_3)(j_4, j_5), \quad (j_2, j_3)(j_5, j_1), \quad (j_3, j_4)(j_5, j_1);$$

(c.3) *a 3-cycle* (k_1, k_2, k_3) *and a product of two disjoint transpositions* $(i_1, i_2)(i_3, i_4)$ *have inner product* $n-3$ *if and only if* $k_1, k_2, k_3 \in \{i_1, i_2, i_3, i_4\}$*;*

(c.4) *two 5-cycles* $(i_1, \ldots, i_5)$ *and* $(j_1, \ldots, j_5)$*, with* $\{i_1, \ldots, i_5\} \neq \{j_1, \ldots, j_5\}$*, have inner product* $n-3$ *if and only if there exists* $k \notin \{i_1, \ldots, i_5\}$ *such that* $(j_1, \ldots, j_5)$ *is one of the following:*

$$(i_1, i_2, i_3, i_4, k), \quad (i_2, i_3, i_4, i_5, k), \quad (i_3, i_4, i_5, i_1, k), \quad (i_4, i_5, i_1, i_2, k), \quad (i_5, i_1, i_2, i_3, k).$$

(d) *Two matrices* $R, S \in \mathbf{A}_n$ *satisfy* $(R, S) = n-4$ *if and only if* $R^t S$ *is the product of 2 disjoint transpositions.*

(e) *Two matrices* $R, S \in \mathbf{A}_n$ *satisfy* $(R, S) = n-5$ *if and only if* $R^t S$ *is a 5-cycle.*

Proof. Let $R, S \in \mathbf{A}_n$. Then computing (R, S) is the same as counting the number of overlapping nonzero positions of the two matrices. Also, it is clear that

$$(R, S) = \operatorname{tr}(RS^t) = (I, R^t S).$$

With these two observations, one immediately get (a), (b), (c), (d), (e).

Suppose we have a 3-cycle (i_1, i_2, i_3) and a 5-cycle $(j_1, \ldots, j_5)$ in $\mathbf{A}_n$. Then the two matrices can overlap at no more than $n-5$ diagonal positions (limited by $(j_1, \ldots, j_5)$), and no more than 2 off-diagonal positions (limited by the number of off-diagonal entries of (i_1, i_2, i_3) that can appear in $(j_1, \ldots, j_5)$). If the two matrices have inner product $n-3$, then both of these upper bounds are attained. So, $\{i_1, i_2, i_3\} \subseteq \{j_1, \ldots, j_5\}$, and two off-diagonal entries of (i_1, i_2, i_3) must appear in $(j_1, \ldots, j_5)$. We get (c.1).

The proofs of (c.2) and (c.3) are similar to that of (c.1). To prove (c.4) let $(i_1, \ldots, i_5)$ and $(j_1, \ldots, j_5)$ be two 5-cycles in $\mathbf{A}_n$ such that $\{i_1, \ldots, i_5\} \neq \{j_1, \ldots, j_5\}$. Then the two matrices can overlap at no more than $n-6$ diagonal positions, and no more than 3 off-diagonal positions. If the two matrices have inner product $n-3$, then both of these upper bounds are attained. Using the fact that the two matrices overlap at 3 off-diagonal positions, one readily gets the conclusion. □

Proof of Theorem 1.4. The sufficiency part is clear. We consider the necessity part. We need only to show that ϕ can be converted to the identity mapping on $\mathbf{V}_n$ by the composite of a sequence of mappings of the form

$$X \mapsto P\phi(X)P^t \qquad \textit{or} \qquad X \mapsto P\phi(X)^t P^t \qquad \text{for some } P \in \mathbf{S}_n. \tag{7}$$

First, we prove the following.

Assertion 1. *Replacing* ϕ *by the composite of a sequence of mappings of the form (7), we can assume that* ϕ *fixes* $I_n, (1,2,3,4,5), (1,2,3),$ *and* $(1,2)(3,4)$*.*

Proof of Assertion 1. Since $\phi(I_n) = I_n$ and (6) is satisfied, it follows from Lemma 2.4 (c) and (e) that ϕ will map 3-cycles to 3-cycles and map 5-cycles to 5-cycles. In particular, we have $\phi((1,2,3,4,5)) = (i_1,\ldots,i_5)$. We may assume that ϕ fixes $(1,2,3,4,5)$; otherwise, let P be a permutation sending $i_1,\ldots,i_5$ to $1,\ldots,5$, respectively, and replace ϕ by the mapping $A \mapsto P\phi(A)P^t$.

Let $X = (1,2,3)$. Then $\phi(X)$ is a 3-cycle. Since ϕ preserves the inner product and fixes $(1,2,3,4,5)$, we have

$$(\phi(X),(1,2,3,4,5)) = (\phi(X),\phi((1,2,3,4,5))) = (X,(1,2,3,4,5)) = n-3.$$

By Lemma 2.4 (c.1),

$$\phi(X) \in \{(1,2,3),(2,3,4),(3,4,5),(4,5,1),(5,1,2)\}.$$

We may assume that $\phi(X) = (1,2,3)$. Otherwise, replace ϕ by the mapping $A \mapsto P\phi(A)P^t$ with

$$P = (1,2,3,4,5)^4, (1,2,3,4,5)^3, (1,2,3,4,5)^2, \text{ or } (1,2,3,4,5),$$

depending on $\phi(X) = (2,3,4),(3,4,5),(4,5,1)$, or $(5,1,2)$, respectively. The resulting map will fix $I_n, (1,2,3,4,5)$, and $(1,2,3)$.

Let $\mathbf{T}$ be the set of products of two disjoint transpositions $R = (i,j)(k,l) \in \mathbf{A}_n$ such that $(R,(1,2,3,4,5)) = n-3$. By Lemma 2.4 (c.2),

$$\mathbf{T} = \{(1,2)(3,4),(1,2)(4,5),(2,3)(4,5),(2,3)(5,1),(3,4)(5,1)\}. \tag{8}$$

Let $X = (1,2)(3,4)$. By Lemma 2.4 (d) and the facts that ϕ fixes I_n and preserves the inner product, we see that $\phi(X)$ is a product of two disjoint transpositions. Since ϕ fixes $(1,2,3,4,5)$ and preserves the inner product, we have

$$(\phi(X),(1,2,3,4,5)) = (X,(1,2,3,4,5)) = n-3,$$

and thus $\phi(X) \in \mathbf{T}$ by the definition of $\mathbf{T}$. Now, since ϕ fixes $(1,2,3)$ and preserves the inner product, we have

$$(\phi(X),(1,2,3)) = (X,(1,2,3)) = n-3.$$

By Lemma 2.4 (c.3) and the fact that $\phi(X) \in \mathbf{T}$, we have $\phi(X) \in \{(1,2)(3,4),(2,3)(1,5)\}$. We may assume that $\phi(X) = (1,2)(3,4)$. Otherwise, let $P = (1,3)(4,5)$ and replace ϕ by the mapping $A \mapsto P\phi(A)^tP^t$. Then the resulting map will fix $I_n, (1,2,3,4,5), (1,2,3)$, and $(1,2)(3,4)$. The proof of Assertion 1 is complete.

Next, we prove an assertion, which will be used repeatedly in the future with $\{1,2,3,4,5\}$ replaced by suitable $\{k_1,k_2,k_3,k_4,k_5\}$.

Assertion 2. *If ϕ fixes $I_n,(1,2,3),(1,2,3,4,5)$, and $(1,2)(3,4)$, then ϕ fixes the matrices in $\mathbf{U}_5$ defined as in (3).*

Proof of Assertion 2. By the argument given in the proof of Assertion 1, we have $\phi(\mathbf{T}) = \mathbf{T}$, where $\mathbf{T}$ is defined as in (8). Moreover, $\mathbf{T}$ may be partitioned into subsets

$$\mathbf{T}_1 = \{(1,2)(3,4),(2,3)(1,5)\},\ \mathbf{T}_2 = \{(1,2)(4,5),(2,3)(4,5)\},\ \text{ and } \mathbf{T}_3 = \{(3,4)(1,5)\},$$

where the elements of $\mathbf{T}_i$ have inner product $n-2-i$ with $(1,2,3)$. Since ϕ fixes $(1,2,3)$ and preserves the inner product, $\phi(\mathbf{T}_i) = \mathbf{T}_i$ for $i = 1,2,3$. So, ϕ fixes $(3,4)(1,5)$. Since ϕ already fixes $(1,2)(3,4)$, ϕ must fix $(2,3)(1,5)$. Finally, since the members of $\mathbf{T}_2$ have different inner products with $(1,2)(3,4)$, ϕ must fix $(1,2)(4,5)$ and $(2,3)(4,5)$. Thus, ϕ fixes each element of

$$\mathbf{F} = \{I_n, (1,2,3,4,5), (1,2,3)\} \cup \mathbf{T}. \tag{9}$$

Next, consider any 3-cycle

$$R = (i_1, i_2, i_3) \qquad \text{with} \qquad i_1, i_2, i_3 \in \{1,2,3,4,5\}. \tag{10}$$

By Lemma 2.4 (c.3), R has inner product $n-3$ with at least one member of $\mathbf{T}$. Let $\phi(R) = (k_1, k_2, k_3)$. Since ϕ fixes each member of $\mathbf{T}$ and preserves the inner product, it follows that (k_1, k_2, k_3) has inner product $n-3$ with at least one member of $\mathbf{T}$, say, with $S = (j_1, j_2)(j_3, j_4)$. Since $(\phi(R), S) = n-3$, it follows from Lemma 2.4 (c.3) that

$$\{k_1, k_2, k_3\} \subseteq \{j_1, j_2, j_3, j_4\} \subseteq \{1,2,3,4,5\}.$$

In other words, ϕ maps any 3-cycle (i_1, i_2, i_3) satisfying (10) to another 3-cycle satisfying (10). Let $X_1, \ldots, X_8$ be the elements in $\mathbf{F}$. For each 3-cycle Y satisfying (10), let

$$v(Y) = ((Y, X_1), \ldots, (Y, X_8)). \tag{11}$$

One can check that if Y_1 and Y_2 are different 3-cycles satisfying (10), then $v(Y_1) \neq v(Y_2)$. (See the Matlab program and output in the Appendix.) Since ϕ preserves the inner product and fixes each element in $\mathbf{F}$, we conclude that ϕ fixes each 3-cycle satisfying (10).

One can check (see the Matlab program at the Appendix) that elements in $\mathbf{F}$ together with the 3-cycles satisfying (10) generate a 17-dimensional subspace, which is the dimension of $\mathbf{U}_5$. Thus, ϕ fixes the elements in a generating set of $\mathbf{U}_5$, and hence it fixes every matrix in $\mathbf{U}_5$. The proof of Assertion 2 is complete.

If $n = 5$, then $\mathbf{U}_5 = \mathbf{V}_n$ and we are done. Suppose $n > 5$. We prove the following assertion.

Assertion 3. *Suppose $5 \le k < n$, and suppose ϕ fixes all the matrices in $\mathbf{U}_k$ defined as in (3). Then one can replace ϕ by a mapping of the form (7) so that the resulting map will fix all matrices in $\mathbf{U}_{k+1}$.*

Proof of Assertion 3. By Proposition 2.1, it suffices to show that ϕ can be modified so that the resulting map will fix all permutations in $\mathbf{A}_n$ of the form $(i_1, i_2)(i_3, i_4)$ with $i_1, i_2, i_3, i_4 \in \{1, \ldots, k+1\}$. Let $X = (1,2,3,4,k+1)$. Then $\phi(X)$ is a 5-cycle by Lemma 2.4 (e). Since ϕ fixes $(1,2,3,4,5)$ and preserves the inner product, we have

$$(\phi(X), (1,2,3,4,5)) = (X, (1,2,3,4,5)) = n-3.$$

By Lemma 2.4 (c.4), $\phi(X)$ equals one of the following:

$$(1,2,3,4,j),(2,3,4,5,j),(3,4,5,1,j),(4,5,1,2,j),(5,1,2,3,j)$$

for some $j > k$. We may assume that $j = k+1$; otherwise, let P be the transposition $(j, k+1)$ and replace ϕ by the mapping $A \mapsto P\phi(A)P^t$. We are going to show that this modified mapping ϕ fixes all matrices in $\mathbf{U}_{k+1}$. Since ϕ also fixes $(1,2)(3,4)$, we have

$$(\phi(X),(1,2)(3,4)) = (X,(1,2)(3,4)) = n-3.$$

By Lemma 2.4 (c.2), it follows that $\phi(1,2,3,4,k+1) = (1,2,3,4,k+1)$. Now, ϕ fixes

$$I_n, \quad (1,2,3), \quad (1,2,3,4,k+1), \quad \text{and} \quad (1,2)(3,4).$$

Using Assertion 2 with 5 replaced by $k+1$, we see that ϕ fixes all permutations in $\mathbf{A}_n$ generated by elements of the form $(j_1,j_2)(j_3,j_4)$ with $j_1,j_2,j_3,j_4 \in \{1,2,3,4,k+1\}$.

Next, we consider a 5-cycle $X = (1,i_1,i_2,i_3,k+1)$ with $i_1,i_2,i_3 \in \{2,\ldots,k\}$. Then $\phi(X)$ is a 5-cycle. Let $i_4 \in \{2,\ldots,k\} \setminus \{i_1,i_2,i_3\}$, $Y_1 = (1,i_1,i_2,i_3,i_4)$, and $Y_2 = (1,i_1)(i_2,i_3)$. Then for $j = 1,2$, we have $\phi(Y_j) = Y_j$ and

$$(\phi(X),Y_j) = (\phi(X),\phi(Y_j)) = (X,Y_j) = n-3. \tag{12}$$

By Lemma 2.4 (c.2) and (c.4), we have $\phi(X) = (1,i_1,i_2,i_3,j)$ for some $j \geq k+1$. If $j > k+1$, then every common nonzero (diagonal, or off-diagonal) position of $(1,2,3,4,k+1)$ and $(1,i_1,i_2,i_3,j)$ is also a common nonzero position of $(1,2,3,4,k+1)$ and $(1,i_1,i_2,i_3,k+1)$. But $(1,2,3,4,k+1)$ and $(1,i_1,i_2,i_3,k+1)$ have common nonzero positions that are not shared by $(1,2,3,4,k+1)$ and $(1,i_1,i_2,i_3,j)$; namely, at the $(1,k+1)$ and (j,j) positions (and also at the $(k+1,4)$ position if $i_3 = 4$). Hence, if $Y_3 = (1,2,3,4,k+1)$, then $(X,Y_3) - (\phi(X),\phi(Y_3)) = 2$ or 3, which is a contradiction. Thus, we must have $j = k+1$ and $\phi(X) = X$. So, ϕ fixes $I_n, (1,i_1,i_2), (1,i_1,i_2,i_3,k+1)$, and $(1,i_1)(i_2,i_3)$. Using Assertion 2 with $(1,2,3,4,5)$ replaced by $(1,i_1,i_2,i_3,k+1)$, we see that ϕ fixes all permutations in $\mathbf{A}_n$ generated by elements of the form $(j_1,j_2)(j_3,j_4)$ with $j_1,j_2,j_3,j_4 \in \{1,i_1,i_2,i_3,k+1\}$.

Suppose $X = (i_1,i_2,i_3,i_4,k+1)$ is a 5-cycle such that $i_j \in \{1,\ldots,k\}$ for $j = 1,\ldots,4$, and $i_1 \neq 1$. Let $i_5 \in \{1,\ldots,k\} \setminus \{i_1,i_2,i_3,i_4\}$, $Y_1 = (i_1,i_2,i_3,i_4,i_5)$, $Y_2 = (i_1,i_2)(i_3,i_4)$, and $Y_3 = (i_2,i_3)(i_1,k+1)$. Note that $\phi(Y_3) = Y_3$ by the result in the preceding paragraph. So, for $j = 1,2,3$, we have $\phi(Y_j) = Y_j$ and (12). By Lemma 2.4 (c.2) and (c.4), we have $\phi(X) = X$. Now, ϕ fixes $I_n, (i_1,i_2,i_3), (i_1,i_2,i_3,i_4,k+1), (i_1,i_2)(i_3,i_4)$. Using Assertion 2 with $(1,2,3,4,5)$ replaced by $(i_1,i_2,i_3,i_4,k+1)$, we see that ϕ fixes all permutations in $\mathbf{A}_n$ generated by elements of the form $(j_1,j_2)(j_3,j_4)$ with $j_1,j_2,j_3,j_4 \in \{i_1,i_2,i_3,i_4,k+1\}$.

Combining the above arguments, we see that the modified mapping ϕ fixes all the permutations in $\mathbf{A}_n$ of the form $(i_1,i_2)(i_3,i_4)$ with $i_1,i_2,i_3,i_4 \in \{1,\ldots,k+1\}$. The proof of Assertion 3 is complete.

Applying Assertion 3 repeatedly, we conclude that ϕ fixes every element in $\mathbf{U}_n = \mathbf{V}_n$. The conclusion of the theorem follows. □

3 Appendix: Matlab Programs and Output

The following Matlab program provides the computational step in the proof of Proposition 2.1, namely, checking that $\mathbf{A}_n \cap \mathbf{U}_4$ contains 10 linearly independent elements. Since every matrix in $\mathbf{A}_n \cap \mathbf{U}_4$ is of the form $P \oplus I_{n-4}$ with $P \in \mathbf{A}_4$, it suffices to show that $\mathbf{A}_4$ has 10 linearly independent elements.

In the program, we first define the standard unit vectors of $\mathbb{R}^4$ in row vector form. Then we express the matrices in $\mathbf{A}_4$ in row vector form and store them in the matrix X. Finally, we apply the "rank" command to check the number of linearly independent row vectors of X. The output "ans = 10" indicates that there are 10 linearly independent elements in $\mathbf{A}_4$, as we claimed.

```
e1= [1 0 0 0]; e2= [0 1 0 0]; e3= [0 0 1 0]; e4= [0 0 0 1];
X = [e1 e2 e3 e4; e2 e1 e4 e3; e3 e4 e1 e2; e4 e3 e2 e1;
     e2 e3 e1 e4; e2 e4 e3 e1; e3 e2 e4 e1; e1 e3 e4 e2;
     e3 e1 e2 e4; e4 e1 e3 e2; e4 e2 e1 e3; e1 e4 e2 e3];
z = rank(X)

ans =

    10
```

The next Matlab program provides two computational steps in the proof of Theorem 1.4, namely,

(i) for $v(Y)$ defined in (11) for a 3-cycle $Y = (i_1, i_2, i_3) \in \mathbf{U}_5$, if Y_1 and Y_2 are two different 3-cycles in $\mathbf{U}_5$ then $v(Y_1) \neq v(Y_2)$,

(ii) the elements in $\mathbf{F}$ defined in (9) and the 3-cycles in $\mathbf{U}_5$ together contain 17 linearly independent matrices.

Since every matrix under consideration is of the form $X_0 \oplus I_{n-5}$, we only need to verify the statement for $n = 5$.

In the following program, we first define the standard unit vectors in $\mathbb{R}^5$ in row vector form. Then we express the eight matrices in $\mathbf{F}$ defined in (9) in the proof of Theorem 1.4 as 1×25 row vectors, and store them as rows in the matrix X. Then we express the twenty 3-cycles in $\mathbf{A}_5$ as 1×25 row vectors, and store them as rows in the matrix Y.

We then compute the $Z = YX^t$. The ith row of this matrix will give

$$v(Y_i) = ((X_1, Y_i), \ldots, (X_8, Y_i))$$

defined as in (11). Then we compare the rows of Z, and check that no two rows are the same, i.e., $v(Y_i) \neq v(Y_j)$ if $Y_i \neq Y_j$. The output "ans = 0 0 " confirms this claim.

Also, we compute the rank of the matrix

$$\begin{bmatrix} X \\ Y \end{bmatrix}$$

and the output "ans = 17" confirms that the matrices in $\mathbf{F}$ and the 3-cycles together generate a 17-dimensional subspace as asserted.

```
e1 = [1 0 0 0 0]; e2 = [0 1 0 0 0]; e3 = [0 0 1 0 0]; e4 = [0 0 0 1 0];
e5 = [0 0 0 0 1];
X = [e1 e2 e3 e4 e5; e2 e3 e1 e4 e5; e2 e1 e4 e3 e5; e5 e3 e2 e4 e1;
     e2 e1 e3 e5 e4; e1 e3 e2 e5 e4; e5 e2 e4 e3 e1; e2 e3 e4 e5 e1];
Y = [e2 e3 e1 e4 e5; e2 e4 e3 e1 e5; e2 e5 e3 e4 e1; e3 e2 e4 e1 e5;
     e3 e2 e5 e4 e1; e4 e2 e3 e5 e1; e1 e3 e4 e2 e5; e1 e3 e5 e4 e2;
     e1 e4 e3 e5 e2; e1 e2 e4 e5 e3; e3 e1 e2 e4 e5; e4 e1 e3 e2 e5;
     e5 e1 e3 e4 e2; e4 e2 e1 e3 e5; e5 e2 e1 e4 e3; e5 e2 e3 e1 e4;
     e1 e4 e2 e3 e5; e1 e5 e2 e4 e3; e1 e5 e3 e2 e4; e1 e2 e5 e3 e4];
Z = Y*X';
z = [0 0];
for r=1:19
  for s=r+1:20
      if Z(r,:) == Z(s,:),
                z = [r,s];
           else
                z = z;
           end
      end
   end
z

ans =

     0     0

rank([X ; Y])

ans =

    17
```

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